On Some Fundamental Concepts of Deductive Theories—I※

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The purpose of the present paper is to define several fundamental concepts of formalised deductive theories, which may eventually be non-constructive, and to give some immediate consequences of the definitions.

The definitions for constructive theories are given in several articles, and, especially, [1][1] is referred to in the manner that the consideration given here is set along the line of the investigation stated in [1].

In this paper the rules of formation are dealt with in the way slightly different from the ordinary one, because it seems convenient for non-constructive theories to obtain a clear concept about "undecidability".

To give more interesting results, those concepts are defined in the formal set theory[2] Σ which is constructed in the first-order predicate logic.

Acquaintance with [2][2] is assumed.

1. Deductive theory.

A deductive theory is determined in terms of the formation of sentences and of the inference, which is the operation on sentences. Sentences represent propositions. Thus, objects of formalised deductive theories are propositions, which are regarded as primitive on account of the generality of the present considerations.

$\boldsymbol{S}$: The set of all meaningful propositions.

The formulas as objects of a deductive system are stipulated by the formation rules, which are defined by the primitive notion $\boldsymbol{S}$. In the following Se is a set expressing $\boldsymbol{S}$ in the set theory.

1.1. Rule. R(X): X is a rule;

$$R_A(X) \equiv (\exists y)(\exists \alpha)(z)(z \in \langle X \rangle \cdot \supset \cdot (\exists u)(\exists v)(z = \langle u, v \rangle \,\&\, u \in A \,\&\, v \subseteq A \,\&\, v \text{ We } y \,\&\, v \simeq \alpha \,\&\, \alpha \in On)).$$

A class X of pairs is called a rule in A, if there are a set y and an ordinal number α such that for every element z of X, the first element of z is an element of

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A and the second element of z is a subset of A that is well-ordered by the set y, and is equivalent to the ordinal number α.

"x We y" means that x is well-ordered by y, and "$x \simeq y$" that x is equivalent to y, on the class of all ordinal numbers. (Cf. [2])

1.2. Decomposability. $Dec_A(X)$.

$Dec_A(X) \equiv (\exists Y)(\overline{\overline{Y}} < \aleph_0 \;\&\; Sum(Y) = X \;\&\; (a)(b)(a \in Y \;\&\; b \in Y \supset E_X(a,b) \vee a = b) \;\&$
$(z)(z \in Y \supset R_A(z))$.

That is, a class X of pairs is called decomposable in A, if X is a sum of finitely many rules, which are mutually exclusive to each other. Sum (Y) means the sum of Y, and $E_X(a,b)$ means that a and b are mutually exclusive.

1.3. Net. Net(X).

A class of pairs is called a net, if the first element of an arbitary element of X, excepting an element called the end-element of X, is included in the second element of at least one element of X.

In symbols,

$Net(X) \equiv (a)(a \in X \supset (\exists u)(\exists v)(a = \langle u, v \rangle)) \;\&\; (\exists b)(c)(\neg b = c \;\&\; c \in W_t(X)$
$\supset c \in Sum(D(X))))$,

where W_t (X) means the domain of the values of X, and D(X) the domain of X.

1.4. End element. $En_Y(x)$

$En_Y(x) \equiv Net(Y) \;\&\; (x \in Wt(Y)) \;\&\; \neg(x \in Sum(D(Y)))$.

An element x of a net Y is called the end element of Y, if x belongs to the range of Y and if x does not belong to the sum of the domain of Y.

1.4.1. *End*(*Y*). The end element of Y.

$x = End(Y) \equiv En_Y(x)$.

1.5. Base. $Bas_Y(X)$.

A class X is called the base of Y, if X is the class of all elements such as are included in the sum of the domain of Y, but not in the range of Y. That is, the base of a rule Y in A, the class of sentences, is the class of all sentences that are previously given and not constructed by the rule Y. In symbols,

$Bas_Y(X) \equiv (z)(z \in X \equiv z \in Sum(D(Y)) \;\&\; \neg(z \in W_t(Y)))$.

$z \in Bas(Y) \equiv (\exists X)(z \in X \;\&\; Bas_Y(X))$.

1.6. Formation rule. $RF_A(X)$.

A class X of pairs is called a formation rule, if X is decomposable in A, a class of sentences, and if its base is denumberable at most. In symbols,

$RF_A(X) \equiv Dec_A(X) \;\&\; \overline{\overline{Bas(X)}} \leq \aleph_0 \;\&\; A \subseteq Se$.

1.7. Formula. $Form_A(X)$.

A class X is called a formula of a formation rule A, if X is a net and $X \subseteq A$.

1.8. $Form_A(X) \equiv RF(A) \;\&\; Net(X) \;\&\; X \subseteq A$.

The class of all formulas. S(A).

$y \in S(A) \equiv Form_A(y)$.

1.9. Rule of inference. $RI_A(X)$, for a formation rule A.

$RI_A(X) \equiv Dec_Y(X) \ \& \ Y \subseteq S(A) \ \& \ Bas(X) \leqq \aleph_0$

1.10. Conseqeuce. $Co_{A,Z}{}^Y(X)$.

$$Co_{A,Z}{}^Y(X) = Y \subseteq S(A) \ \& \ RI_A(z) \ \& \ (a)(a \in X. \equiv . A \in Y \ v (\exists u)$$
$$(Net(u) \ \& \ u \subseteq z \ \& \ End_u(a) \ Bas(u) \subseteq Y)$$

A class X of formulas is called the consequence of Y for a rule of inference z, if, for any element of x, either a is included in Y or there exists a net $u \subseteq z$ such that Bas $(u) \subseteq Y$ and the end-element of u is a.

1.10.1. Class of consequences. $Con_{A,Z}(Y)$.

$$x \in Con_{A,Z}(Y) \equiv (\exists u)(Con_{A,Z}{}^Y(u) \ \& \ x \in u)$$

1.11. Deductive theory. DT ((X, Y)).

The ordered pair (X, Y) is called a deductive theory if there is a class A such that RF (X) and RI (Y).

2. From the above definitions the following theorems are immediately obtained, corresponding to [1].

2.1. $X \subseteq S(A) \ \& \ RI_A(Y) \supset (X \subseteq Con_{A,Y}(X) \subseteq S(A))$.

2.2. $X \subseteq S(A) \ \& \ RI_A(Y) \supset (Con_{A,Y}(Con_{A,Y}(X)) = (Con_{A,Y}(X))$.

2.3. $X+Y \subseteq S(A) \ \& \ RI_A(Z) \supset Con_{A,Z}(X+Y) = Con_{A,Z}(X+Con_{A,Z}(Y))$
$= Con_{A,Z}(Con_{A,Z}(X)+Con_{A,Z}(Y))$.

2.4. $X \subseteq Y \subseteq S(A) \ \& \ RI_A(Z) \supset Con_{A,Z}(X) \subseteq Con_{A,Z}(Y)$.

2.5. $A \subseteq S(B) \ \& \ RI_B(X) \ \& \ RI_B(Y) \ \& \ X \subseteq Y \supset Con_{B,X}(A) \subseteq Con_{B,Y}(A)$.

2.6. $A \subseteq S(B) \ \& \ RI_B(X) \ \& \ RI_B(Y) \ \& \ X \subseteq Y \supset Con_{B,X}(Con_{B,Y}(A))$
$= Con_{B,Y}(Con_{B,Y}(A)) = Con_{B,Y}(A)$.

2.7. $A \subseteq S(B) \ \& \ RI_B(X) \ \& \ RI_B(Y) \supset (Con_{B,X}(Con_{B,Y}(A))$
$= Con_{B,Y}(A) \supset (Con_{B,X}(A) \subseteq Con_{B,Y}(A)))$

2.8. $A \subseteq S(B) \ \& \ RI_B(X) \ \& \ RI_B(Y) \supset (Con_{B,X}(Con_{B,Y}(A))$
$= Con_{B,X}(A) \supset Con_{B,Y}(A) \subseteq Con_{B,X}(A))$.

3. Standard deductive theory.

To obtain non-trivial results, other restrictions are added to the class S(A) and the rule of inference.

3.1. Section. $Sect_Y(X)$.

$$Sect_y(X) \equiv Net(X) \ \& \ Net(Y) \ \& \ End(X)$$
$$= End(Y) \ \& \ Sum(D(X)) \subseteq W_t(Y) \ \& \neg(X=Y)$$

A net X is called a section of a net Y, if the end-element of X is the end-element of Y and if X is properly included in Y, and every element of the sum of the domain of X is an element of the range of Y.

3.2. Single-domain, SD(X).

$$SD(X) \equiv (a)(\exists Y)(\exists b)(y = \langle a,b \rangle \& \ y \in X \supset (z)(u)$$
$$(z \in X \ \& \ z = \langle a,u \rangle \supset b=u)$$

A class X of ordered pairs is called single-domained if for any element a there exists at most one second element b such that $\langle a,b\rangle \in X$.

3.3. Component of a rule, $Copn_Y(X)$.

$$Copn_Y(X) \equiv R(X) \ \& \ X \subseteq Y \ \& \ Dec_A(Y) \ \& \ (\exists A)(\overline{\overline{A}} < \aleph_0 \ \& (z)(z \in A \supset (\exists B)(R_B(z)))$$
$$Sum\ (A) = Y \& \ (a)(b)(a \in A \ \& \ b \in A \supset E_X(a,b))$$

A class X of ordered pairs is called a component of Y, if X is a rule included in Y, and if Y is composable into finitely many rules z_i exclusive mutually, one of which is X.

3.4. ${}_XH_Y$.

A class X of ordered pairs is called homogeneous to a class Y of ordered pairs if there exists a one-to-one correspondence C between X and Y that satisfies the following.

3.4.0.1. An element a of X that belongs to a rule f corresponds to an element $C'a \in Y$ that belongs to the same rule f, and vice versa.

3.4.0.2. If a α—th element b in the second element of an element a of X is the first element of an element of a rule g, then the α—th element $C'b$ in the second element of the element $C'a$ of Y is the first element of an element of the same element g.

In symbols,

3.4.1. O_X.

$$\langle a,b\rangle \in O_X \equiv Un_Z(O_X) \& D(O_X) = \alpha \& W_t\ (O_X) = X \& a \in X \& b < \alpha \& Y \simeq \alpha.$$

3.4.2. ${}_XH_Y$.

$${}_XH_Y \equiv (\exists z)(Un_Z(Z) \ \& \ D(Z) = X \ \& \ W(Z) = Y \& (a)(\exists b)(R(b) \& (a \in X \ \& \ a \in b$$
$$\equiv Z'a \in Y \ \& \ Z'a \in b) \ \& \ (a)(\exists b)(R(b) \ \& \ (a \in Y \ \& \ a \in b \equiv Z^{-1\prime}a \in X \& Z^{-1\prime}a \in b)).$$
$$(a)(u)(v)(t)(s)(\beta)(a \in X \& a = \langle u,v\rangle \in b \in Y \& b = \langle t,s\rangle \& \langle b,a\rangle \in Z$$
$$\supset (f)(\beta)(R(f)\beta \in On \supset Ov'\beta \in f \equiv Os'\beta \in f) \ \& \ Os'\beta, Ov'\beta\rangle \in Z^{-1})$$

3.5. Segment. ${}_XSeg_Y$.

$${}_xSeg_Y = (\exists z)(zH_X \& z \subseteq Y)$$

That is, a class of ordered pairs is called a segment od Y, if there exists a subclass z of Y such that X is homogeneous to Y.

3.6. Same scope. XSSY.

A class X is called having the same scope as a class Y, if either X is a segment of Y or Y is a segment of X.

$$XSSY = {}_XSeg_Y . \vee . {}_YSeg_X.$$

3.7. A deductive theory (X, Y) is called standard if the following conditions 3.71.—3.7.14 are satisfied. In symbols, NS((X, Y)).

3.7.1. FL(X):

$$FL(X) \equiv (A)(\alpha)(\exists\beta)(x)(a)(b)$$
$$(Copn_X(A) \ \& \ X \in A \ \& \ X = \langle a,b\rangle \cdot \supset \cdot O_b{}'\alpha SSB \& Net(B)).$$

3.7.2. $CLD_Y(X)$:

$$CLD_Y(X) \equiv \neg(\exists A)(X \subseteq A \ \& \ R_Y(A) \ \& \ FL(A)).$$

3.7.3. SD(X):

3.7.4. X has a component z consisting of the elements whose first element is of the form$\sim$a and whose second element is of the forma. In symbols,

$$(\exists z)(\mathrm{Copn}_X(z) \& (a)(b)(\langle a,b\rangle \in X \supset (\exists u)(\{u\}=b \;\&\; a=\sim u)),$$

where$\sim$a means the negation of the sentence a and then in the set theory $\Sigma\sim a$ is a set of Σ.

3.7.5. X has a component z consisting of the elements, the second element of which is {a,b}, and the first element of which is a$\supset$b, where naturally$\supset$in a$\supset$b is not the logical symbol that the author used above.

$$(\exists z)(\mathrm{Copn}_X(z) \& (a)(b)(\langle a,\ b\rangle \in X \supset (\exists u)(\exists v)(b=\{u,v\} \& a=u\supset v))).$$

3.7.6. There exists a formula $a\subseteq S(X)$ such that $\mathrm{Con}_{X,Y}(X+\{a\})=S(X)$.
In symbols, $(\exists a)(\mathrm{Con}_{X,Y}(X+\{a\}=S(X))$.

3.7.7. $U\leqq S(X) \;\&\; a\in S(X) \;\&\; a\supset b\in \mathrm{Con}_{X,Y}(U) \cdot\supset\cdot b\in \mathrm{Con}_{X,Y}(U+\{a\})$.

3.7.8. $z\subseteq S(X) \;\&\; a\in S(X) \;\&\; b\in S(X) \;\&\; b\in \mathrm{Con}_{X,Y}(z+\{a\}) \cdot\supset\cdot (a\supset b\in \mathrm{Con}_{X,Y}(z))$.

3.7.9. $a\in S(X) \cdot\supset\cdot \mathrm{Con}_{X,Y}(\{a,\sim a\})=S(X)$.

3.7.10. $(z)\{a\in S(X)\supset \mathrm{Con}_{X,Y}(z+\{a\}).\{\mathrm{Con}_{X,Y}(z+\{\sim a\})=\mathrm{Con}_Y(z)\}$.

3.7.11. FL(X). FL(Y).

3.7.12. $\mathrm{CLD}_S(X)$. $\mathrm{CLD}_X(Y)$.

3.7.13. $a\in W_t\ (Y) \cdot\supset\cdot (\exists z)(\mathrm{Net}(z) \;\&\; z\subseteq Y \;\&\; \overline{\overline{z}}<\aleph_0 \;\&\; W_t\ (Y) \;\&\; \mathrm{Bas}(z)\subseteq \mathrm{Bas}(Y)$.

3.7.14. Every component of Y consists of sach elements as follows:

3.7.14.1. All the first e element have the same segment.

3.7.14.2 For any ordinal number α, all the α-th elements of the second elements have the same segment.

$\mathrm{Copn}_Y(z)\supset(\alpha)(\exists s)(\exists t)(u)(a)(b)(u\in z \;\&\; u=\langle a,\ b\rangle\supset t$ seg as Seg $O_b{}'\alpha)$.

3.8. Consistency.

$$WiF_{A,Y}(X)\equiv \mathrm{Con}_{A,Y}(X)\subseteq S(A) \;\&\; \neg(\mathrm{Con}_{A,Y}(X)=S(A)) \& X\subseteq A.$$

That is, the class X of formulas in A is called consistent for a rule of inference Y, if the class of the consequences of X is a proper subclass of S(A).

3.9. Completeness.

$$Compl_{A,Y}(X)\equiv(z)(\mathrm{WiF}_{A,Y}(z) \& X\subseteq z \& \neg(X=z) \cdot\supset\cdot \mathrm{Con}_{A,Y}(z)=S(A).$$

3.10. From the above definitions in 3, the following is easily shown. Assume that NS((X, Y)).

$$\mathrm{RI}_X(Y) \& x\in S(X) \;\&\; y\in S(X) \& z\in S(X) \;\&\; \mathrm{NS}((X,Y)) \cdot\supset\cdot ((x\supset y)\supset((y\supset z)$$
$$\supset(x\supset z)))\in \mathrm{Con}_{X,Y}(O),$$ where O is the empty set.

3.10.2. $(z)(z\in Y\supset \mathrm{RI}_X(z) \;\&\; \mathrm{NS}((x,z))) \;\&\; ((z)(z\in Y\supset U\subseteq Z)$
$\& (v)((z)(z\in Y\supset V\subseteq z)\supset v\subseteq U)\supset \mathrm{RI}_X(U) \;\&\; \mathrm{NS}((X,\ U)))$.

If U is the inter-section of a class of the rules of in X, then (X, V) is a standard theory.

3.10.3. $((B)(B\in A\supset \mathrm{RI}_X(B) \& \mathrm{NS}((X,B))) \& \mathrm{RI}_X(\mathrm{Sum}(A)) \& \mathrm{NS}((X,\mathrm{Sum}(A))))$
$\supset \mathrm{Sum}(A)\in A$.

That is, there is no class Y, of rules of inference, which contains at least two different rules, and which itself is a rule.

3.10.4. $WiF_{X,Y}(A) \equiv A \subseteq S(X) \& \neg(\exists x)(x \in Con_{X,Y}(A) \& \sim x \in Con_{X,Y}(A))$

3.10.5. $Compl_{X,Y}(A) \equiv (x)(x \in S(X) \supset x \in Con_{X,Y}(A) \& \sim x \in Con_{X,Y}(A))$.

3.10.6. $Compl_{X,Y}(A) \supset \neg(\exists B)(Wif,_{X,Y}(B) \& Com_{X,Y}(B) \& A \subseteq B \& \neg A = B)$.

References

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