On a Relativization of Quantifiers※

By

Kempachiro Ōhashi

0. A. Tarski in [1][1)] investigated a relativization of quantifiers applicable to prove various undecidabilities, which seems to be the standard type of the kind.

A trivial extension of the relativization in [1] is obtained as follows; For any formula F of a theory T, denoting by F° the formula given by replacing every subformula of F, (x)G(x), or ($\exists$x)G(x), with (x)(P(t,x)$\supset$G(x)), or ($\exists$x)(P(t,x) & G(x)), respectively, provided that t is a variable not contained in F. Let F^{*} be

$$(P(t,a_1)\supset(\cdots\cdots(P(t,a_n)\supset F^{\circ}(a_1,\cdots, a_n))\cdots),$$

for all free variables a_i of F. (t)F(t) is denoted by $F^{(P)}$.

Then, theorem 9 and 10 in [1], 1.5, hold for the above relativization.

In this paper another relativization will be considered, that is in use to prove the independence of an axiomatic number theory in [2][2)].

0.1. The relativization to be considered here is defined as follows; For any formula F of a theory and a binary predicate P(t,x), the formula F° is obtained by replacing every subformula of the form, (x) G (x), or ($\exists$x) G (x), by a formula (x)(P(t,x)$\supset$ G (x)), or ($\exists$x)(P(t,x) & G (x)), respectively, provided that t, which is called the sub-variable of F, is not contained in F. If F contains free variables $a_1, \cdots, a_m$, and individual constant terms, $c_1,\cdots, c_n$, F is of the following form;

$$(P(t,a_1)\supset(P(t,a_2)\supset(\cdots(P(t,c_1)\supset(\cdots(P(t,c_n)\supset F^{\circ}))\cdots)$$

where t is the sub-variable of F.

(F^{*}, correlated with the formula F containing neither free variable nor individual constant, is F°.) Let ($\exists$s) (t) (P (s,t) $\supset$ F^{*}) be denoted by $F^{(P)}$, where s is not contained in F and t is the sub-variable of F. Since the sets of free variables and individual constant terms can be well ordered by a suitable ordering defined in advance, the transformation from F to $F^{(P)}$ is uniquely determined. Without lossing generality, it is assumed that all formulas F have the same sub-variable.

0.2. Theory $T^{(P)}$.

0.2.1. Constants of $T^{(P)}$ are those of T or of the predicate P(.,.).

0.2.2. The set of all formulas valid in $T^{(P)}$ is the intersection of all the sets including any formula $F^{(P)}$ correlated with a formula F valid in T, and is closed

※ Contribution from the Shimonoseki College of Fisheries No. 281, Received Aug. 5, 1959.

under application of the rules of inference in T.

1. From the definition of $T^{(P)}$, we have the following theorem.

Theorem 1. For any theory and for any binary predicate P(.,.), which is not contained in T, adding the following axioms to $T^{(P)}$

(y)(∃x)P(y,x), (u)(v)(x)(P(u,v)⊃(P(v,x))), (u)(v)((x)(P(u,x)⊃(P(v,x) (x)sP∨,x)⊃ P(u,x))

1.1. $T^{(P)}$ is axiomatizable if and only if T is axiomatizable.

1.2. If T contains only finitely many individual constants and operation symbols, $T^{(P)}$ is finitely axiomatizable if and only if T is finitely axiomatizable.

Proof.

1.3. Assume that T is axiomatizable. There is a recursive set S of valid formulas of T from which every formula valid in T is derivable. Let S′ be the set of the following formulas;

1.3.1. all the formulas $F^{(P)}$ correlated with the formulas F of S.

1.3.2. the formulas, (y)(∃x) P (y,x), (u)(v)(x) P ((u,v)⊃(P (v,x)⊃ P (u,x))) (u)(v) ((x)(P (u,x)⊃ P (v,x)). ∨. (x)(P (v,x)⊃ P (u,x)))

1.3.3. all the formulas of the forms,

(y) P (y,c), (y)(x)(P (y,x)⊃ P (y,sx)),⋯,

$(y)(x_1)\cdots(x_n)(P(y,x_1)\supset(\cdots(P(y,x_n)\supset P(y,f_n(x_1,\cdots,x_n))\cdots),\cdots$, etc.,

where c is an individual constant, s a unary operation symbol, f_n (⋯⋯) n-ary operation symbol,⋯ etc.

1.3.4. From the above definition S′ is recursive, and every formulas in S′ is valid in T.

1.3.5. If F is a formula of S, then $F^{(P)}$ is derivable from S′.

1.3.7. Suppose that $F^{(P)}$ and $(F\supset G)^{(P)}$ are derivable from S′.

(∃u)(t)(P (u,t)⊃F*) and (∃v) (t)(P (v,t)⊃(F⊃ G)*) are derivable from S′. From 1.3.2, the following formula is derivable from S′.

((t)(P (u,t)⊃F*) & (t)(P (v,t)⊃(F⊃ G)*)⊃(t)(P (v,t)⊃F* & (F⊃ G)*)) ∨. ((t) (P (u,t)⊃F*) & (t)(P (v,t)⊃(F⊃ G)*)⊃(t)(P (u,t)⊃F* & (F⊃ G)*)) where u and v are neither in F*, nor (F⊃G)*. Then,

(t) (P (u,t)⊃F*) & (t) (P (v,t)⊃(F⊃ G)*)
⊃(∃u) (t) (P (u,t)⊃F* & (F⊃G)*)

is derivable. Therefore (∃u)(t)(P (u,t) ⊃ F* & (F ⊃ G)*) is derivable from S′. Finally, (∃u) (t) (P (u,t)⊃ G*) is derivable from S′, that is, $G^{(P)}$ is derivable from S′.

1.3.8. We consistently assume that the rule of detachment is the only rule of inference.[3] Hence, every valid formula of $T^{(P)}$ is derivable from S′.

Thus the axiomatizability of $T^{(P)}$ is established.

1.4. Suppose that $T^{(P)}$ is axiomatizable. There is a recursive set M′ of formulas valid in $T^{(P)}$ from which every formula valid in $T^{(P)}$ is derivable. Let M be the set of all the formulas F*, which is obtained by replacing P(x,y) by x=x & y=y in the formulas $F^{(P)}$ of M', and of $(x)(x=x)$.

1.4.1. Given any formula F valid in T, the correlated formula $F^{(P)}$ is derivable from M', and, moreover, if a formula $F^{(P)}$ is derivable from M', then by the corresponding process, which is given by replacing $P(x, y)$ by $x=x$ & $y=y$, the correlated formula* F is derivable from M.

Thus, any formula valid in T is derivable from M, since $(x)(x=x)$ and* F implies F.

1.4.2. If $A^{(P)}$ is derivable from the set of the formulas $B_i{}^{(P)}$ correlated with the set of the formulas B_i valid in T, then the formula *A correlated with $A^{(P)}$ is derivable from the set of the formulas $*B_i$ correlated with $B^{(P)}$. Hence, if A is valid in T, then the correlated formula *A is derivable from the set of the formulas *F correlated with the valid formulas $F^{(P)}$ valid in $T^{(P)}$, which are in use to imply $A^{(P)}$, corresponding to A. From the above definition, as *F is valid in T, *A is valid in T. Thus, any formula derivable from M is valid in T.

From 1.4.1 and 1.4.2, the set of all formulas derivable from M is the set of all formulas valid in T. We have the axiomatizability of T.

In 1.2, if S is finite, S' is finite, and M is finite for M' which is finite. Therefore, 1.2 holds. Thus, Theorem 1 is obtained.

2. Theorem 2. For any theory T and $P(.,.)$ a binary predicate not contained in T,

$T^{(P)}$ is essentially undecidable if and only if T is essentially undecidable, provided that $(x)(y)(P(x,y) \vee P(y,x))$ and $(x)(y)(z)(P(x,y) \supset (P(y,z) \supset P(x,z)))$ are valid in $T^{(P)}$.

Proof. From further consideration along the line of 1.4, it is easily seen that $T^{(P)}$ is interpretable in T, and that the consistency of T implies that of $T^{(P)}$.

Now, let T be inconsistent. Then two formulas A and $\neg A$ are valid in T and, $A^{(P)}$ and $(\neg A)^{(P)}$ are valid in T.

$(\neg A)^{(P)}$ is of the following form,

$$(x)(t)(P(x,t) \supset (\neg A)^*).$$

From the validity of the formula $(x)(y)(P(x,y) \vee P(y,x))$ and $(x)(y)(z)(P(x,y) \supset (P(y,z) \supset P(x,z))$, $(x)(\exists t)(P(x,t)$ & $\neg(A^*))$ is derivable in $T^{(P)}$.

Thus, $A^{(P)}$ and $\neg(A^{(P)})$ being valid in $T^{(P)}$, $T^{(P)}$ is inconsistent. We have the proposition,

2.1.1. $T^{(P)}$ is consistent if and only if T is consistent.

2.2. Moreover, from 2.1, similarly to [1], p. 28--29, it is shown that, $T^{(P)}$ is essentially undecidable if and only if T is essentially undecidable.

Thus,

Theorem 2 is proved.

References

1) [1]. TARSKI, A. (1953) Uudecidable theories. (In collaboration with A. MOSTOWSKI and R. M. ROBINSON.)

2) [2]. OHASHI, K. (1959) On ω-incompleteness of an axiomatic number theory. to appear. *J. Shimonoseki Coll. of Fisheries, Natural Scie.* No. 4.

3) Cf. [1], 1. 2.