On a Problem of Elementary Number Theory※

By

Kempatiro ŌHASI

§ 1 Introduction

The purpose of this paper is to present a solution of the following problem in [I][1] (p. 279, footnote) ; Is the formula (A) provable or independent in a logical system of first order with axioms, (J_1), (J_2), $(<_1)$, $(<_2)$, $(<_3)$, and the axiom of induction?

(A) $\quad a<b \supset a'=b \vee a'<b$

B_1:

(J_1) $\quad a=a$

(J_2) $\quad a=b \supset (A(a) \supset A(b))$

$(<_1)$ $\quad \neg a<a$

$(<_2)$ $\quad a<b \,\&\, b<c \supset a<c$

$(<_3)$ $\quad a<a'$

(I) $\quad A(o) \,\&\, (x)(A(x) \supset (A(x'))\supset A(a)$

The formula (A), as Hilbert and Bernays pointed out in [I], is provable in the system with bound predicate variables, and its proof is as follows (they didn't show it in [I].)

1.1 Proof of $\quad a<b \equiv (\exists F)\ ((x)(F(x) \supset F(x'))\,\&\,\neg F(a)\,\&\,F(b))$

1.1.1 Definition $\quad F_o(c) \overset{\text{D.f.}}{\equiv} a<c$

1.1.2 $F_o(c) \supset F_o(c')$

1.1.3 $a<b \supset (x)\ (F(x) \supset F(x'))$ $\quad$ 1.1.2

1.1.4 $a<b \supset F_o(b)$ $\quad$ 1.1.1

1.1.5 $a<b \supset F(a)$ $\quad$ $(<_1)$

1.1.6 $a<b \supset (x)\ (F_o(x) \supset F_o(x'))\ \&\neg F_o(a)\,\&\,F_o(b)$ $\quad$ 1.1.3, 1.1.4, 1.1.5

1.1.7 $a<b \supset (\exists F)\ (x)\ (F(x) \supset F(x'))\ \&\neg F(a)\ \&F(b))$ $\quad$ 1.1.6

1.1.8 $F(o)\ \&\ (x)(F(x) \supset F(x')) \supset F(a)$ $\quad$ (I)

1.1.9 $\neg(\exists F)((x)(F(x) \supset F(x'))\,\&\neg F(a)\,\&\,F(o))$ $\quad$ 1.1.8

1.1.10 $(\exists F)\ ((x)(F(x) \supset F(x'))\,\&\,F(a)\,\&\,F(o)) \supset a<o$ $\quad$ 1.1.9

1.1.11 Definition $\quad P(b) \overset{\text{D.f.}}{\equiv} (\exists F)((x)(F(x) \supset F(x'))\,\&\neg F(a)\,\&\,F(b)) \supset a<b$

1.1.12 $P(o)$

※ Contribution from the Shimonoseki College of Fisheries No. 249

1.1.13 $(\exists F)\ ((x)(F(x)\supset F(x'))\&\neg F(\mathrm{a})\&F(\mathrm{b}'))$
$\supset(x)(F_1(x)\supset F_1(x'))\&\neg F_1(\mathrm{a})\&F_1(\mathrm{b}'))$

1.1.14 Definition $F_2(\mathrm{c})\overset{\mathrm{D.f.}}{\equiv}F_1(\mathrm{c})\vee \mathrm{c}=\mathrm{b}$

1.1.15 $F_1(\mathrm{b}')\supset F_2(\mathrm{b})$ 1.1.14

1.1.16 $F_1(\mathrm{b}')\supset(x=\mathrm{b}\supset F_1(x'))$ (J_2)

1.1.17 $F_1(\mathrm{b}')\supset(x=\mathrm{b}\supset F_2(x'))$ 1.1.16

1.1.18 $(x)\ (F_1(x)\supset F_1(x'))\supset(F_1(x)\supset F_2(x'))$ 1.1.14

1.1.19 $F_1(\mathrm{b}')\&(x)(F_1(x)\supset F_1(x'))\supset(x)(F_2(x)F_2(x'))$ 1.1.17, 1.1.18

1.1.20 $(\exists F)((x)(F(x)\supset F(x'))\&\neg F(\mathrm{a})\&F(\mathrm{b}'))$
$\supset(x)(F_2(x)\supset F_2(x'))\&\neg F_1(\mathrm{a})\&F_2(\mathrm{b})$ 1.1.13, 1.1.15, 1.1.19

1.1.21 $(x)(F_2(x)\supset F_2(x'))\&\neg F_1(\mathrm{a})\&F_2(\mathrm{b})$
$\supset((x)(F_2(x)\supset F_2(x'))\&\neg F_2(\mathrm{a})\&F_2(\mathrm{b}))\vee \mathrm{a}=\mathrm{b}$ 1.1.14

1.1.22 $(\exists F)((x)(F(x)\supset F(x'))\&\neg F(\mathrm{a})\&F(\mathrm{b}'))$
$\supset(\exists F)((x)(F(x)\supset F(x'))\&\neg F(\mathrm{a})\&F(\mathrm{b}))\vee \mathrm{a}=\mathrm{b}$ 1.1.20, 1.1.21

1.1.23 $P(\mathrm{b})\supset P(\mathrm{b}')$ 1.1.21, $(<_3)$

1.1.24 $P(\mathrm{o})\&(x)(P(x)\supset P(x'))$ 1.1.12, 1.1.23

1.1.25 $P(\mathrm{b})$ 1.1.24

1.1.26 $\mathrm{a}<\mathrm{b}\equiv(\exists F)((x)(F(x)\supset F(x'))\&\neg F(\mathrm{a})\&F(\mathrm{b}))$ 1.1.7, 1.1.25

1.2 Proof of (A)

1.2.1 $\mathrm{a}<\mathrm{b}'\supset(x)(F_3(x)\supset F_3(x'))\&\neg F_3(\mathrm{a})\&F_3(\mathrm{b}')$

1.2.2 Definition $F_4(\mathrm{c})\overset{\mathrm{D.f.}}{\equiv}F_3(\mathrm{c})\vee \mathrm{c}=\mathrm{b}$

1.2.3 $F_3(\mathrm{b}')\supset F_4(\mathrm{b})$ 1.2.2

1.2.4 $(x)(F_3(x)\supset F_3(x'))\supset(F_3(x)\supset F_4(x'))$ 1.2.2

1.2.5 $F_3(\mathrm{b}')\supset(x=\mathrm{b}\supset F_4(x'))$ (J_2), 1.2.2

1.2.6 $(x)(F_3(x)\supset F_3(x'))\&F(\mathrm{b}')\supset(x)(F_4(x)\supset F_4(x'))$ 1.2.4, 1.2.5

1.2.7 $\mathrm{a}<\mathrm{b}'\supset(x)(F_4(x)\supset F_4(x'))\&\neg F_3(\mathrm{a})\&F_4(\mathrm{b})$ 1.2.1, 1.2.3

1.2.8 $\mathrm{a}<\mathrm{b}'\supset((x)(F_4(x)\supset F_4(x'))\&\neg F_4(\mathrm{a})\&F_4(\mathrm{b}))\vee \mathrm{a}=\mathrm{b}$ 1.2.2, 1.2.7

1.2.9 $\mathrm{a}<\mathrm{b}'\supset(\exists F)((x)(F(x)\supset F(x'))\&\neg F(\mathrm{a})\&F(\mathrm{b}))\vee \mathrm{a}=\mathrm{b}$ 1.2.8

1.2.10 $\mathrm{a}<\mathrm{b}'\supset \mathrm{a}=\mathrm{b}\vee \mathrm{a}<\mathrm{b}$ 1.2.9

1.2.11 $\mathrm{a}<\mathrm{b}\supset(x)(F_5(x)\supset F_5(x'))\&\neg F_5(\mathrm{a})\&F_5(\mathrm{b})$ 1.1.26

1.2.12 Definition $F_6(\mathrm{c})\overset{\mathrm{D.f.}}{\equiv}F_5(\mathrm{c})\&\mathrm{b}<\mathrm{c}$

1.2.13 $\mathrm{a}<\mathrm{b}\supset(x)(F_6)(x)F_6(x'))\&\neg F_5(\mathrm{a})\&F_6(\mathrm{b}')$ 1.2.11, 1.2.12

1.2.14 $\mathrm{b}<\mathrm{a}'\supset(\mathrm{b}=\mathrm{a}\vee \mathrm{b}<\mathrm{a})$ 1.2.10

1.2.15 $\mathrm{a}<\mathrm{b}\supset(\mathrm{b}<\mathrm{a}'\supset \mathrm{a}\supset \mathrm{a})$ 1.2.14, $(<_2)$

1.2.16 $\mathrm{a}<\mathrm{b}\supset\neg\mathrm{b}<\mathrm{a}'$ 1.2.15, $(<_1)$

1.2.17 $\mathrm{a}<\mathrm{b}\supset(x)(F_6(x)\supset F_6(x'))\&(\neg F_5(\mathrm{a}')\vee\neg \mathrm{b}<\mathrm{a}')\ \&F_6(\mathrm{b}')$
1.2.13, 1.2.16

1.2.18 $\mathrm{a}<\mathrm{b}\supset(\mathrm{E}F)((x)(F(x)\supset F(x'))\&\neg F(\mathrm{a}')\ \&F(\mathrm{b}'))$ 1.2.17

1.2.19 $\mathrm{a}<\mathrm{b}\supset \mathrm{a}'<\mathrm{b}'$ 1.2.18, 1.1.26

1.2.20 $a < b' \supset a' = b' \vee a' < b'$ 1.2.10

1.2.21 Definition $F(a, b) \overset{\text{D.f.}}{=} a < b \supset a' = b \vee a' < b$

1.2.22 $F(a, b) \supset F(a, b')$ 1.2.20

1.2.23 $F(a, o)$ 1.2.21

1.2.24 $F(a, o) \& (x)(F(a, x) \supset F(a, x'))$ 1.2.22, 1.2.23

1.2.25 $a < b \supset a' = b \vee a' < b$ 1.2.24, 1.2.21

From the above proof we naturally agree with Hilbert and Bernays in the point that the independence of the formula (A), as they suggested in [I], can be proved only by close investigations into the deductive formalism, provided that (A) is independent in the system. For, it is conditioned by limitations of our formalism.

The independence-proofs of such formulas, which seem to lie on the border line of formulas, are sometimes succeeded by uses of Godel's theorem[2] that sufficiently strong formal systems cannot prove their own consistency ; but in the present case it seems difficult to find any suitable relation between our problem and Godel theorem, and hence, we don't adopt that method. The method used here is one by examining formal proofs step by step, for which Gentzen's system L. K [3] is convenient. In this paper we shall consider our problem in the formal system consisting of LK, the inference of induction and several axioms, which is easily proved to be equivalent to the above system.

§ 2 applies to the exposition of the underlying logical system used later.

In § 3 a theorem concerning to the axiom of induction will be proved. In § 4 the independence of (A) and its cosequence will be proved.

§ 2 Logical system and the inference-figure of induction.

The inference-schemata listed in this § are due to L K in [2].

2.1 Symbols.

2.1.1 Constants.

2.1.1.1 A particular individual. 0

2.1.1.2 A function. $'$

2.1.1.3 Predicates ,

2.1.1.4 Logical symbols

$\&,\ v,\ \neg,\ \supset,\ \equiv,\ (x),\ (\exists x)$

2.1.2 Variables.

2.1.2.1 Individual variables

a, b, c, ..., x, y, z, ...

2.1.2.2 Predicate variables.

A, B, C, ...

2.2 Notions.

We shall use a, b, ············A, B, ············ as metamathematical symbols too, since no confusion is likely to occur.

2.2.1 Terms.

2.2.1.1 Individual constant and individual variables are terms.

2.2.1.2 If a is a term, then so is a$'$.

2.2.1.3 The only terms are those given by 2.2.1.1 and 2.2.1.2

2.2.2 Formulas.

2.2.2.1 If A is a predicate and t_1 and t_2 are terms, then $A(t_1, t_2)$ is a formula.

2.2.2.2 If A and B are formulas, then

$$A \& B,\ A \vee B,\ \neg A,\ A \supset B,\ A \equiv B,\ (x)A(x),\ (\exists x)A(x) \text{ are formulas.}$$

2.2.2.3 The only formulas are those given by 2.2.2.1 and 2.2.2.2.

2.2.3 Sequents.

2.2.3.1 If A_1, ·········, A_m, B_1, ·········, B_n are formulas, then

$$A_1, \cdots\cdots, A_m \longrightarrow B_1, \cdots\cdots, B_n$$

is called a "sequent."

2.2.3.2 In 2.2.3.1, A_1, ············, A_m are called "the antecedent" of the sequent, and B_1, ············, B_n are called "the succedent" of the sequent.

2.2.4 Inference-figures.

If A_1, ············A_n, B are sequents, then

$$\frac{A_1, \cdots\cdots A_n}{B}$$

is called "a inference-figure", and A_1, ·········, A_n are called "the upper sequents" of the inference-figure, B "the lower sequent".

2.2.5 Proof-figures.

2.2.5.1 Every sequent in a proof-figure is an upper sequent of an inference-figure, except the sequent called "the end-sequent".

2.2.5.2 Every sequent in a proof-figure is an upper sequent of at most one inference-figure.

2.2.5.3 In any proof-figure there is no sequent such as arrivable again to its own sequent in tracing sequents to its lower sequents.

2.2.5.4 Sequents in a proof-figure, which are not lower sequents of any sequent, are called "the beginning sequents" in the proof-figure, where beginning sequents must have such a form as $A \rightarrow A$, or be an axiom.

2.3 Inference-schemata.

2.3.1 Inference-schemata on structure of sequents.

"Thinning"

in antecedent $\dfrac{\Gamma \rightarrow \triangle}{A,\ \Gamma \rightarrow \triangle}$ in succedent $\dfrac{\Gamma \rightarrow \triangle}{\Gamma \rightarrow \triangle,\ A}$

"Contraction"

in antecedent $\dfrac{A,\ A,\ \Gamma \rightarrow \triangle}{A,\ \Gamma \rightarrow \triangle}$ in succedent $\dfrac{\Gamma \rightarrow \triangle,\ A,\ A}{\Gamma \rightarrow \triangle,\ A}$

"Interchange"

in antecedent $\dfrac{\Delta,\ A,\ B,\ \Gamma \to \Lambda}{\Delta,\ B,\ A,\ \Gamma \to \Lambda}$ in succedent $\dfrac{\Gamma \to \Delta,\ B,\ A,\ \Lambda}{\Gamma \to \Lambda,\ A,\ B,\ \Lambda}$

In these inference-figures A and B in the upper sequent are called "the sub-formulas" of the inference-figure, and A, B in the lower sequent are called "the chief-formulas" of the inference-figure.

2.3.2 "Cut"

$$\frac{\Gamma \to \Delta,\ A \qquad A,\ \pi \to \Lambda}{\Gamma,\ \pi \to \Delta,\ \Lambda}$$

2.3.3 Inference-schemata on logical symbols.

US $\dfrac{\Gamma \to \Delta,\ A \qquad \Gamma \to \Delta,\ B}{\Gamma \to \Delta,\ A \& B}$ UA (1) $\dfrac{A,\ \Gamma \to \Delta}{A \& B,\ \Gamma \to \Delta}$

(2) $\dfrac{B,\ \Gamma \to \Delta}{A \& B,\ \Gamma \to \Delta}$

OS (1) $\dfrac{\Gamma \to \Delta,\ A}{\Gamma \to \Delta,\ A \vee B}$ OA $\dfrac{A,\ \Gamma \to \Delta \qquad B,\ \Gamma \to \Delta}{A \vee B,\ \Gamma \to \Delta}$

(2) $\dfrac{\Gamma \to \Delta,\ B}{\Gamma \to \Delta,\ A \vee B}$

AS $\dfrac{\Gamma \to \Delta,\ A(a)}{\Gamma \to \Delta,\ (x)A(x)}$ AA $\dfrac{A(t),\ \Gamma \to \Delta}{(x)A(x),\ \Gamma \to \Delta}$

where a is a free variable not contained in the lower sequent. a is called the eigen-variables of this inference-figure.

where t is an arbitrary variable.

ES $\dfrac{\Gamma \to \Delta,\ A(t)}{\Gamma \to \Delta,\ (\exists x)A(x)}$ EA $\dfrac{A(a),\ \Gamma \to \Delta}{(\exists x)A(x),\ \Gamma \to \Delta}$

where t is an arbitrary term.

where a is a free variable not contained in the lower sequent. a is called the eigen-variable of this inference-figure.

NS $\dfrac{A,\ \Gamma \to \Delta}{\Gamma \to \Delta,\ \neg A}$ NA $\dfrac{\Gamma \to \Delta,\ A}{\neg A,\ \Gamma \to \Delta}$

FS $\dfrac{A,\ \Gamma \to \Delta,\ B}{\Gamma \to \Delta,\ A \supset B}$ FA $\dfrac{\Gamma \to \Delta,\ A \qquad B,\ \pi \to \Lambda}{A \supset B,\ \Gamma,\ \pi \to \Delta,\ \Lambda}$

In the above schemata, the formulas denoted by A, B, $A(a)$ or $A(t)$ in the upper sequent are called "the sub-formulas" of the inference-figure, and the formulas denoted by A, $A \& B$, $A \vee B$, $\neg A$, $(x)A(x)$ or $(\exists x)A(x)$ in the lower sequent are called "the chief-formulas" of the inference-figure.

2.4 Inference-schema of induction.

$$\frac{\Gamma \to \Delta,\ A(o) \qquad A(a),\ \Gamma \to \Delta,\ A(a')}{\Gamma \to \Delta,\ A(t)}$$

Where t is an arbitrary term, and a is a free variable not contained in the lower sequent. a is called the eigen-variable of this inference-figure. $A(a)$; $A(a')$ are called "the sub-formulas" of the inference-figure, and $A(t)$ is called "the chief-formula" of the inference-figure.

2.5 We shall consider a formal system with the inference-figures 2.3, 2.4 henceforce. Evidently, adding (J'_1), (J'_2), $(<'_1)$, $(<'_2)$, $(>'_3)$ as the axioms to that system, we obtain a system equivalent to the system for which our problem arises.

$$B'\begin{cases} (J'_1) & \rightarrow a = a \\ (J'_2) & \rightarrow a = b \supset (A(a) \supset A(b)) \\ (<'_1) & \rightarrow \neg\, a < a \\ (<'_2) & \rightarrow a < b \;\&\; b < c \supset a < c \\ (<'_3) & \rightarrow a < a' \end{cases}$$

2.6 There is another neat inference-schema (I') equivalent to 2.4

$$(\mathrm{I})\quad \frac{F(a),\ \Gamma \rightarrow \Delta,\ F(a')}{F(o),\ \Gamma \rightarrow \Delta,\ F(t)}$$

The equivalence is proved as follows ;

$$\frac{\Gamma \rightarrow \Delta,\ F(o) \qquad \dfrac{F(a),\ \Gamma \rightarrow \Delta,\ F(a')}{F(o),\ \Gamma \rightarrow \Delta,\ F(t)}}{\Gamma \rightarrow \Delta,\ F(t)}$$

$$\frac{\dfrac{F(o) \rightarrow F(o)}{\vdots\ \downarrow}\qquad\qquad \dfrac{F(a),\ \Gamma \rightarrow \Delta,\ F(a')}{F(o),\ F(a),\ \Gamma \rightarrow \Delta,\ F(a')}}{F(o),\ \Gamma \rightarrow \Delta,\ F(o) \qquad F(a),\ F(o),\ \Gamma \rightarrow \Delta,\ F(a')}$$

$$\frac{F(o),\ \Gamma \rightarrow \Delta,\ F(o) \qquad F(a),\ F(o),\ \Gamma \rightarrow \Delta,\ F(a')}{F(o),\ \Gamma \rightarrow \Delta,\ F(t)}$$

But we take the unrefined form 2.4, because this is suitable for our purpose.

2.7 Let LK_o be a system consisting of $L\,K$ and B' , and $L\,K_I$ be a system $L\,K_o$ +2.4.

§ 3 A theorem on the inference-schema of induction.

3.1 A sequent S is called "provable" if there is a proof-figure, the end-sequent of which is S, and a formula A is called "provable" if there is a provable sequent of the form $\rightarrow A$.

3.2 A proof-figure P is called to be "reduced" to a proof-figure Q, if the following conditions are satisfied.

3.2.1 The end-sequent of P is equivalent to the end-sequent of Q, in the sence that both of those sequents are provable in $L\ K_1$ or neither of them is provable in $L\ K_1$.

3.2.2 In Q, there is no inference-figure of the form 2.3.2, 2.3.3 below any inference of the form 2.4, excepting the inferences of AS, ES, and equivalent transformations with respect to the quantifiers of the formulas in the end-sequent.

3.3 In 3.2, Q is called "the normal proof" of P, and it is called " a reduction" to give a normal proof.

3.4 In this §, the following theorem will be proved.

Theorem. 1. Let a normal formula A be of the form

$$(x_1)\ (x_n)\ F(x_1 \cdots\cdots\cdots\cdots, x_n).$$

If A is provable, then there is a normal proof of A.

By "normal" is here meant that all the quantifiers stand at the beginning, with no negatives between or in front of them, and have scopes extending to the end of the formula.

3.5 In the following, we will prove this theorem by the mathematical induction on the grade, which is the sum of the numbers of 2.3.2, 2.3.3 contained in the proof-figure, and hence our theorem will be proved if we can give a reduction for each i dividual inference-figure of 2.3.2, 2.3.3 in P. We may assume thereby, that there is only one inference-figure of 2.4 above the given inference-figure.

We shall mainly give only reductions without full explanations of it, for reductions are given sufficiently in detail.

3.6 If the variable t of the chief-formula of a inference of 2.4 is a constant, we have an equivalent proof without the inference of 2.4. It can be seen as follows ; Any constant can be written $O^{(n)}$ $(O^{(n)} = O,\ O',O'', \cdots\cdots\cdots\cdots)$

3.6.1 The case, $n=1$.

$$\dfrac{\Gamma \to \triangle,\ A(o) \qquad \dfrac{\dfrac{A(a),\ \Gamma \to \triangle,\ A(a')}{\vdots}}{A(o),\ \Gamma \to \triangle,\ A(o')}}{\dfrac{\dfrac{\Gamma,\ \Gamma \to \triangle,\ \triangle,\ A(o')}{\vdots}}{\Gamma \to \triangle,\ A(o')}}$$

3.6.2 Suppose that for $n=k$ the above property is true.

$$\dfrac{\dfrac{\Gamma \to \triangle,\ A(o) \qquad A(a),\ \Gamma \to \triangle,\ A(a')}{\Gamma \to \triangle,\ A(o^{(k)})} \qquad \dfrac{\dfrac{A(a),\ \Gamma \to \triangle,\ A(a')}{\vdots}}{A(o^{(k)}),\ \Gamma \to \triangle,\ A(o^{(k+1)})}}{\dfrac{\dfrac{\Gamma,\ \Gamma \to \triangle,\ \triangle,\ A(o^{(k+1)})}{\vdots}}{\Gamma \to \triangle,\ A(o^{(k+1)})}}$$

Thus the above property is true for arbitrary n. Therefore we need consider only the cases where t is a variable term.

Reductions.

3.7 The case, where the inference-figure R is a cut, can be divided into two cases, 3.7.1 and 3.7.2.

3.7.1 The chief-formula of 2.4 is not the sub-formula of R, as follows ; 3.7.1.1—2.

3.7.1.1 in left

$$\dfrac{\dfrac{\dfrac{\Gamma\to\Delta, B, A(o) \qquad A(a), \Gamma\to\Delta, B, A(a')}{\Gamma\to\Delta, B, A(t)}}{\Gamma\to\Delta, A(t), B} \qquad\qquad B, \pi\to\Lambda}{\Gamma, \pi\to\Delta, A(t), \Lambda}$$

3.7.1.2 in right

$$\dfrac{\Gamma\to\Delta, B \qquad\qquad \dfrac{B, \pi\to\Lambda, A(o) \qquad A(a), B, \pi\to\Lambda, A(a')}{B, \pi\to\Lambda, A(t)}}{\Gamma, \pi\to\Delta, \Lambda, A(t)}$$

3.7.1.3 "$\overline{\quad\vdots\quad}$" stands for a proof-figure without 2.4, which is easily seen, or a well known method of changing the free variables.

Reduction of 3.7.1.1

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma\to\Delta, B, A(o)}{\Gamma\to\Delta, A(o), B} \qquad B, \pi\to\Lambda}{\Gamma, \pi\to\Delta, A(o), \Lambda}}{\dfrac{\vdots}{\Gamma, \pi\to\Delta, \Lambda, A(o)}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a), \Gamma\to\Delta, B, A(a')}{\text{b is a variable not contained in } \Gamma, \Delta, \pi, \Lambda, A(a)}}{A(b), \Gamma\to\Delta, B, A(b')}}{A(b), \Gamma\to\Delta, A(b'), B} \qquad B, \pi\to\Lambda}{A(b), \Gamma, \pi\to\Delta, A(b'), \Lambda}}{\dfrac{\vdots}{A(b), \Gamma, \pi\to\Delta, \Lambda, A(b')}}}{\dfrac{\Gamma, \pi\to\Delta, \Lambda, A(t)}{\dfrac{\vdots}{\Gamma, \pi\to\Delta, A(t), \Lambda}}}$$

Reduction of 3.7.1.2

$$\dfrac{\dfrac{\Gamma\to\Delta, B \qquad B, \pi\to\Lambda, A(o)}{\Gamma, \pi\to\Delta, \Lambda, A(o)} \qquad \dfrac{\dfrac{\Gamma\to\Delta, B \qquad \dfrac{\dfrac{\dfrac{A(a), B, \pi\to\Lambda, A(a')}{\text{b is a variable not contained in } \rceil, \Delta, \pi, \Lambda, A(a)}}{A(b), B, \pi\to\Lambda, (b')}}{B, A(b), \pi\to\Lambda, A(b')}}{\Gamma, A(b), \pi\to\Delta, \Lambda, A(b')}}{\dfrac{\vdots}{A(b) \text{ a } \Gamma, \pi\to\Delta, \Lambda, A(b')}}}{\Gamma, \pi\to\Delta, \Lambda, A(t)}$$

3.7.2 The chief-formula of 2.4 is the sub-formula of R.

$$\dfrac{\dfrac{\Gamma\to\Delta, A(o), \qquad A(a), \Gamma\to\Delta, A(a')}{\Gamma\to\Delta, A(t)} \qquad\qquad A(t), \pi\to\Lambda}{\Gamma, \pi\to\Delta, \Lambda}$$

3.7.2.1 Either Δ or Λ has elements.

Suppose Δ is not empty, and let Δ be Δ^*, B.

Reduction is as follows.

$$A(b),\ \Gamma\to\triangle,\ A(b')$$

$$A(b),\ \Gamma\to\triangle,\ A(b') \qquad\qquad A(t),\ \pi\to V$$

$$A(b),\ \Gamma,\ \pi\to\triangle,\ \Lambda,\ A(b') \qquad \pi\to\Lambda,\ \neg A(t)$$

$$A(b),\ \Gamma,\ \pi\to\triangle,\ \Lambda,\ \neg A(t)$$

$$\Gamma\to\triangle,\ A(o) \qquad A(t),\ \pi\to\Lambda \qquad B\to B \qquad A(b),\ \Gamma,\ \pi\to\triangle,\ \Lambda,\ A(b')\&\neg A(t)$$

$$B\to(A(b')\&\neg A(t))YB$$

$$A(b),\ \Gamma,\ \pi\to\triangle^*,\ \Lambda,\ (A(b')\&\neg A(t))\vee B$$

$$B\Gamma,\ \pi\to\triangle^*,\Lambda,\ (A(b')\&\neg A(t))\wedge B \qquad (A(b)\&\neg A(t,\ \Gamma,\ \pi\to\triangle^*\Lambda,\ (A(b')\&\neg A(t))\vee B$$

$$\Gamma,\ \pi\to\triangle^*,\ \Lambda,\ (A(o)\&\neg A(t))\vee B \qquad (A(b)\&\neg A(t))\vee B,\ \Gamma,\ \pi\to\triangle^*,\Lambda,\ ((b')\&\neg A(t))\vee B$$

$$\Gamma,\ \pi\to\triangle^*,\Lambda,\ (A(t)\&\neg A(t)(\vee B$$

$$\Gamma,\ \pi\to\triangle^*,\ (A(t)\&\neg A(t))\vee B,\ \Lambda$$

$\Gamma,\ \pi\to\triangle^*,\ (A(t)\&\neg A(t))\vee B,\ \Lambda$ is equivalent to $\Gamma,\ \pi\to\triangle,\ \Lambda$

3.7.2.2 $\triangle$and Λ both are empty.

$$A(a),\ \Gamma\to A(a')$$

$$A(t),\ \pi\to$$

$$\Gamma\to A(o) \qquad \pi\to\neg A(o) \qquad A(b),\ \Gamma\to A(b') \qquad \pi\to\neg A(t)$$

$$\Gamma,\ \pi\to A(o) \qquad \Gamma,\ \pi\to\neg A(t) \qquad A(b),\ \Gamma,\ \pi\to A(b') \qquad A(b),\ \Gamma,\ \pi\to\neg A(t)$$

$$\Gamma,\ \pi\to A(o)\&\neg A(t) \qquad A(b),\ \Gamma,\ \pi\to(A(b')\&\neg A(t))$$

$$\Gamma,\ \pi\to A(t)\&\neg A(t)$$

$\Gamma,\ \pi\to A(t)\&\neg A(t)$ is equivalent to $\Gamma,\ \pi\to$

3.8 The case, where R is a inference-figure of US, can be divided into four cases, 3.8.1—4.

3.8.1 The sub-formulas of R is not a chief-formula of 2.4.

Reduction is analogous to 3.7.1.

3.8.2 The left sub-formula of R is a chief-formula of 2.4.

$$\dfrac{\dfrac{\Gamma\to\triangle,\ A(o)\qquad A(a),\ \Gamma\to\triangle,\ A(a')}{\Gamma\to\triangle,\ A(t)}\qquad \Gamma\to\triangle,\ B}{\Gamma\to\triangle,\ A(t)\&B}$$

Reduction.

$$\dfrac{\dfrac{\Gamma\to\triangle,\ A(o)\qquad \Gamma\to\triangle,\ B}{\Gamma\to\triangle,\ A(o)\&B}\qquad \dfrac{\dfrac{A(a),\ \Gamma\to\triangle,\ A(a')\qquad \dfrac{\Gamma\to\triangle,\ B}{A\ (a)\ \Gamma\to\triangle A\ (a)}}{A(a),\Gamma\to\triangle,\ A(a')\&B}}{A(a)\&B,\ \Gamma\to\triangle,\ A(a')\&B}}{\Gamma\to\triangle,\ A(t)\&B}$$

3.8.3 The right sub-formula of R is a chief-formula of 2.4.

Reduction is analogous to 3.8.2.

3.8.4 Both sub-formulas of R are chief-formulas of 2.4.

$$\dfrac{\dfrac{\Gamma\to\triangle,\ A(o)\qquad A(a),\ \Gamma\to\triangle,\ A(a')}{\Gamma\to\triangle,\ A(t)}\qquad \dfrac{\Gamma\to\triangle,\ B(o)\qquad B(b),\ \Gamma\to\triangle,\ B(b')}{\Gamma\to\triangle,\ B(s)}}{\Gamma\to\triangle,\ A(t)\&B(s)}$$

Reduction is as follows.

3.8.4.1

$$\dfrac{\dfrac{\Gamma\to\triangle,\ A(o)\qquad \Gamma\to\triangle,\ B(o)}{\Gamma\to\triangle,\ A(o)\&B(o)}\qquad \dfrac{\dfrac{A(a),\ \Gamma\to\triangle,\ A(a')\qquad \dfrac{\dfrac{\Gamma\to\triangle,\ B(o)}{\vdots}}{A(a),\ \Gamma\to\triangle,\ B(o)}}{A(a),\ \Gamma\to\triangle,\ A(a')\&B(o)}}{A(a)\&B(o),\ \Gamma\to\triangle,\ A(a')\&B(o)}}{\Gamma\to\triangle,\ A(t)\&B(o)}$$

3.8.4.2

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma\to\triangle,A(o)}{B(d),\Gamma\to\triangle,A(o)}\qquad \dfrac{\dfrac{B(b),\Gamma\to\triangle,B(t)}{\vdots}}{B(d),\Gamma\to\triangle,B(d')}}{B(d),\ \Gamma\to\triangle,\ A(o)\&B(d')}}{A(t)\&B(d),\Gamma\to\triangle,A(o)\&B(d)}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a),\Gamma\to\triangle,A(a')}{\vdots}}{A(c),\Gamma\to\triangle,A(c')}}{A(c),\&B(d),\Gamma\to\triangle,A(c')}\qquad \dfrac{\dfrac{\dfrac{B(b'),\Gamma\to\triangle,B(b')}{\vdots}}{B(d'),\Gamma\to\triangle,B(d')}}{A(c)\&B(d'),\Gamma\to\triangle,B(d')}}{A(c)\&B(d'),\Gamma\to\triangle,A(c')\&B(d')}}{A(c)\&B(d'),A(t)\&B(d),\Gamma\to\triangle,A(c')\&B(d')}}{A(t)\&B(d),\ \Gamma\to\triangle,\ A(t)\&B(d')}$$

3.8.4.3

$$\dfrac{3.8.4.1 \qquad\qquad 3.8.4.2}{\Gamma\to\Delta,\ A(t)\&B(s)}$$

Where c is not included in $B(d)$, Γ, Δ, and d is not included in $A(t)$, Γ, Δ.

3.9 The case UA. There is only one possibility such as follows ;

$$\dfrac{\dfrac{A,\ \Gamma\to\Delta,\ D(o) \qquad D(a),\ A,\ \Gamma\to\Delta,\ D(a')}{A,\ \Gamma\to\Delta,\ D(t)}}{A\&B,\ \Gamma\to\Delta,\ D(t)}$$

Reduction ; let b be a variable not contained in A, B, $D(a)$, Γ, Δ.

$$\dfrac{\dfrac{A,\ \Gamma\to\Delta,\ D(o)}{A\&B,\ \Gamma\to\Delta,\ D(o)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{D(a),\ A,\ \Gamma\to\Delta,\ D(a')}{\vdots}}{D(b),\ A,\ \Gamma\to\Delta,\ D(b')}}{\dfrac{A,\ D(b),\ \Gamma\to\Delta,\ D(b')}{A\&B,\ D(b),\ \Gamma\to\Delta,\ D(b')}}}{D(b),\ A\&B,\ \Gamma\to\Delta,\ D(b')}}{A\&B,\ \Gamma\to\Delta,\ D(t)}$$

3.10 The case of OS is divided into two cases, 3.9.1—2.

We shall consider only the case (1).

3.10.1 The sub-formula of R is not the chief-formula of 2.4.

This case is trivial.

3.10.2 The sub-formula of R is the chief-formula of 2.4.

$$\dfrac{\dfrac{\Gamma\to\Delta,\ A(o) \qquad A(a),\ \Gamma\to\Delta,\ A(a')}{\Gamma\to\Delta,\ A(t)}}{\Gamma\to\Delta,\ A(t)\vee B}$$

Reduction ; let b be a variable not contained in Γ, Δ, $A(a)B$.

$$\dfrac{\dfrac{\Gamma\to\Delta,\ A(o)}{\Gamma\to\Delta,\ A(o)\vee B} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A(a),\Gamma\to\Delta,A(a')}{\vdots}}{A(b),\Gamma\to\Delta,A(b')}}{A(b),\Gamma\to\Delta.A(b')\vee B} \qquad \dfrac{\dfrac{\dfrac{B\to B}{\vdots}}{B,\ \Gamma\to\Delta,\ B}}{B,\Gamma\to\Delta,A(b')\vee B}}{A(b)\vee B,\ \Gamma\to\Delta,\ A(b')\vee B}}{\Gamma\to\Delta,\ A(t)\vee B}$$

3.11 The case AS can be dividwed in two cases, 3.11.1—2.

3.11.1 The sub-formula of R is not a chief-formula of 2.4.

$$\dfrac{\dfrac{\dfrac{\Gamma\rightarrow\triangle,\ B(\mathrm{b}),\ A(\mathrm{o}) \qquad A(\mathrm{a}),\ \Gamma\rightarrow\triangle,\ B(\mathrm{b}),\ A(\mathrm{a}')}{\Gamma\rightarrow\triangle,\ B(\mathrm{b}),\ A(t)}}{\Gamma\rightarrow\triangle,\ A(t),\ B(\mathrm{b})}}{\Gamma\rightarrow\triangle,\ A(t),\ (x)B(x)}$$

Reduction is easily given because b is not included in Γ, $\triangle$, $A(t)$.

3.11.2 The sub-formula of R is a chief-formula of 2.4.

$$\dfrac{\dfrac{\Gamma\rightarrow\triangle,\ A(\mathrm{o}) \qquad A(\mathrm{a}),\ \Gamma\rightarrow\triangle,\ A(\mathrm{a}')}{\Gamma\rightarrow\triangle,\ A(\mathrm{b})}}{\Gamma\rightarrow\triangle,\ (x)A(x)}$$

Two cases arise ; 3.11.2.1—2.

3.11.2.1 If $(x)A(x)$ remained in the end-sequent, then the end-sequent must be of the form

$$\rightarrow (x) \cdots A(x,\ \cdots)$$

and the proof-figure from the sequent in problem to the end-sequent is as follows ;

$$\begin{array}{c} \Gamma\rightarrow\triangle,\ (x)A(x) \\ \hline \\ \downarrow \\ \\ \hline \rightarrow\cdots(x)A(x,\cdots) \end{array}$$

Since neither Γ nor $\triangle$ contains the variable a and $(x)A(x)$ is operated only by US, and then $(x)A(x)$ has no variable contained in any other formula of the above proof-figure. Therefore, taking suitable variables b, c, we obtain a reduction. It needs consider only the inference of 2.4, which is in problem now, and then it is enough to present the following proof-figure.

$$\dfrac{\dfrac{\begin{array}{c} \\ \\ \\ \downarrow \\ \hline \rightarrow A(\mathrm{o}) \end{array} \qquad \begin{array}{c} A(\mathrm{a}),\ \Gamma\rightarrow\triangle,\ A(\mathrm{a}') \\ \hline \vdots \\ \hline A(\mathrm{b}),\ \Gamma\rightarrow\triangle,\ A(\mathrm{b}') \\ \hline \downarrow \\ \hline A(\mathrm{b})\rightarrow\cdots A(\mathrm{b}') \end{array}}{\rightarrow\cdots A(\mathrm{c})}}{\rightarrow(x)\cdots A(x,\ \cdots)}$$

From the condition of the theorem,

$\rightarrow\cdots(x)A(x,\ \cdots)$ is equivaleut to $(x)\cdots A(x,\ \cdots)$

3.11.2.2 $(x)A(x)$ is eliminated in a place, as follows ;

3.11.2.2.1

$$\dfrac{\begin{array}{c} \Gamma\to\triangle,\ A(a) \\ \hline \Gamma\to\triangle,\ (x)A(x) \\ \hline \downarrow \\ \hline \Gamma_1 \to \triangle_1,\ B \end{array} \qquad \begin{array}{c} \downarrow \\ \hline B,\ \pi_1 \to \Lambda_1 \end{array}}{\Gamma_1,\ \pi_1 \to \triangle_1,\ \Lambda_1}$$

where B is generated from $(x)A(x)$.

3.11 2.2.2

$$\dfrac{\begin{array}{c} \downarrow \\ \hline \pi_1 \to \Lambda_1,\ B \end{array} \qquad \begin{array}{c} \Gamma\to\triangle,\ A(a) \\ \hline \Gamma\to\triangle,\ (x)A(x) \\ \hline \downarrow \\ \hline B,\ \Gamma_1 \to \triangle_1 \end{array}}{\pi_1,\ \Gamma_1 \to \Lambda_1,\ \triangle_1}$$

B is generated fom $(x)A(x)$.

Reduction of 3.11.2.2.1.

Four cases arise, 3.112.2.1.1—4.

3.11.2.2.1.1

In the proof of $B,\ \pi_1 \to \Lambda_1$, the right upper sequent in 3.11.2.2.1, $(x)\ A(x)$ represents, by 2.3.1, as follows.

$$\begin{array}{c} \pi_0 \to \Lambda_0 \\ \hline (x)A(x),\ \pi_0 \to \Lambda_0 \\ \hline \downarrow \\ \hline (x)A(x),\ \pi \to \Lambda \\ \hline \downarrow \\ \hline B,\ \pi_1 \to \Lambda_1 \end{array}$$

Reduction.

$$\begin{array}{c} \pi_0 \to \Lambda_0 \\ \hline \downarrow \\ \hline \pi_1 \to \Lambda_1 \\ \hline \downarrow \\ \hline \Gamma_1,\ \pi_1 \to \triangle_1,\ \Lambda_1 \end{array}$$

3.11.2.2.1.2 In the proof of $B,\ \pi \rightarrow \Lambda,\ (x)A(x) \rightarrow (x)A(x)$ is a biginning sequent.

$$\underline{(x)A(x) \rightarrow (x)A(x)}$$
$$\downarrow$$
$$\overline{\underline{(x)A(x),\ \pi_0 \rightarrow \Lambda_0,\ (x)A(x)}}$$
$$\downarrow$$
$$\overline{B,\ \pi_1 \rightarrow \Lambda_1,\ \mathrm{c}}$$

where B is generated from $(x)\ A\ (x)$ in the antecedent, and C is generated from $(x)A(x)$ in the succedent.

Reduction.

$$\underline{\Gamma \rightarrow \Delta,\ (x)A(x)}$$
$$\vdots$$
$$\overline{\underline{\Gamma,\ \pi^* \rightarrow \Delta,\ \Lambda^*,\ (x)A(x)}}$$
$$\downarrow$$
$$\overline{\Gamma,\ \pi \rightarrow \Delta,\ \Lambda\ \Lambda\ \mathrm{c}}$$

3.11.2.2.1.3 In the proof of $B, \pi_1 \rightarrow \Lambda_1, (x)A(x)$ is derived from $A(\mathrm{a})$ by UA.

$$\frac{A(\mathrm{b}),\ \pi \rightarrow \Lambda}{(x)\ A\ (x),\ \pi \rightarrow \Lambda}$$
$$\downarrow$$
$$\overline{B,\ \pi_1 \rightarrow \Lambda_1}$$

Reduction.

$$\dfrac{\Gamma \rightarrow \Delta,\ A(\mathrm{a}) \qquad \dfrac{\dfrac{A(\mathrm{b}),\ \pi \rightarrow \Lambda}{\pi^*,\ \Lambda^* \text{ has no variable common in } F,\ \Delta,}}{A(\mathrm{a}),\ \pi^* \rightarrow \Lambda^*}}{\Gamma,\ \pi^* \rightarrow \Delta,\ \Lambda^*}$$

$$\dfrac{\dfrac{\dfrac{\Gamma,\ \pi^* \rightarrow \Delta,\ \Lambda^*}{\Gamma,\ \pi^* \rightarrow \Delta,\ \Lambda^*,\ (x)A(x)}}{\downarrow}}{\dfrac{\Gamma_1,\ \pi^* \rightarrow \Delta_1,\ \Lambda^*,\ B}{\dfrac{\downarrow}{\Gamma_1,\ \pi^*,\ \rightarrow \Delta_1,\ \Lambda_1{}^*,\ B}}} \qquad \dfrac{\dfrac{\dfrac{\Gamma,\ \pi^* \rightarrow \Delta,\ \Lambda^*}{(x)A(x),\ \Gamma,\ \pi^* \rightarrow \Delta,\ \Lambda^*}}{\downarrow}}{\dfrac{B,\ \Gamma,\ \pi_1{}^* \rightarrow \Delta,\ \Lambda_1{}^*}{\dfrac{\downarrow}{B,\ \Gamma^1,\ \pi_1{}^* \rightarrow \Delta_1,\ \Lambda_1{}^*}}}$$

$$\overline{\underline{\Gamma_1,\ \pi_1{}^*,\ \Gamma_1,\ \pi_1{}^* \rightarrow \Delta_1,\ \Lambda_1{}^*,\ \Delta_1,\ \Lambda_1{}^*}}$$
$$\vdots$$
$$\overline{\underline{\Gamma_1,\ \pi_1{}^* \rightarrow \Delta_1,\ \Lambda_1{}^*}}$$
$$\vdots \qquad (\mathrm{a})$$
$$\overline{\Gamma_1,\ \pi_1 \rightarrow \Delta_1,\ \Lambda_1}$$

3.11.2.2.1.4. The formula generated from $(x)A(x)$ is derived from a formula independent of $(x)A(x)$.

$$\begin{array}{c} B_0,\ \pi_0 \to \Lambda_0 \\ \hline B_0 \& B_1,\ \pi_0 \to \Lambda_0 \\ \downarrow \\ \hline B,\ \pi_1 \to \Lambda_1 \end{array}$$

In this case, in the proof of $\Gamma_1 \to \Delta_1,\ B$, there is the following proof-figure.

$$\begin{array}{c} \Gamma_0 \to \Delta_0,\ B_0 \qquad \Gamma_0 \to \Delta_0,\ B_1 \\ \hline \Gamma_0 \to \Delta_0,\ B_0 \& B_1 \\ \downarrow \\ \hline \Gamma_1 \to \Delta_1,\ B \end{array}$$

Therefore, we have a reduction, as follows ;

$$\begin{array}{c} \Gamma_0 \to \Delta_0,\ B_0 \qquad B_0,\ \pi_0 \to \Lambda_0 \\ \hline \Gamma_0,\ \pi_0 \to \Delta_0,\ \Lambda_0 \\ \downarrow \\ \hline \Gamma_1,\ \pi_1 \to \Delta_1,\ \Lambda_1 \end{array}$$

3.11.2.2.1.5 In the proof of $B,\ \pi_1 \to \Lambda_1$ there is a formula, which contains $(x)\ A(x)$ but is not generated from it.

$$\begin{array}{c} B_1,\ \pi \to \Lambda \\ \hline \downarrow \\ \hline B,\ \pi_1 \to \Lambda_1 \end{array}$$

In this case $B_1 \to B_1$ is a biginning sequent or is derived in the figure by 2.3.1, "Thinning".
The former case is analogous to 3.11.2.2.1.2, and the case "thinning" is analogous to 3.11.2.2.1.1.

In the above reductions, since the eigen-variables are eliminated in the lower sequents, when the variable-conditions really affect, the variable-changes in (a) are guaranteed. For, Γ, π, Δ, Λ in any inference of 2.3.3 are not changed when the sub-formulas aer operated, and in the inference 2.3.2, $(x)A(x)$ is not operated.
Thus we have a normal proof of the inference UA.
The case 3.11.2.2.2 is analogous to 3.11.2.2.1.
3.12 the case EA, 3.13 the case AA, 3.14 the case ES can be discussed analogously.
3.15 In the case NS, it is analogous to 3.10.
3.16 The case, where R is one of NA, is divided into two cases, 3.16.1—2.

3.16.1 The sub-formula of R is not the chief-formula of 3.2.4. So the reduction is easy.

3.16.2 The following case, where the sub-formula of R is the chief-formula of 2.4, is not so simple.

$$\dfrac{\dfrac{\Gamma\to\Delta,\ A(\mathrm{o}) \qquad A(\mathrm{a}),\ \Gamma\to\Delta,\ A(\mathrm{a}')}{\Gamma\to\Delta,\ A(t)}}{\neg A(t),\ \Gamma\to\Delta}$$

In this case there is likely no simple reduction, hence we shall consider how $A(t)$ is treated thereafter. There are eight possibilities, cut, OA, UA, EA, AA, NS, FS, FA, since $A(t)$ is not contained in antecedent.

3.16.2.1 "cut"

$$\dfrac{\Gamma\to\Delta,\ \neg A(t) \qquad \neg A(t),\ \pi\to\Lambda}{\Gamma,\ \pi\to\Delta,\ \Lambda}$$

Assume that t is $\mathrm{b}^{(n)}$. Let $A^*(\mathrm{b})$ be $A(\mathrm{b}^{(n)})$.

Reduction is as follows ;

$$\dfrac{\dfrac{\overline{\pi\to\Lambda,\ A(\mathrm{o})} \qquad \overline{\Gamma\to\Delta,\ \neg A(t)}}{\vdots}\ \ \Gamma,\ \pi\to\Delta,\ \Lambda,\ A(\mathrm{c})\&(A(t) \qquad \dfrac{\overline{\Gamma\to\Delta,\ \neg A(t)} \qquad \overline{A(\mathrm{a}),\ \pi\to\Lambda,\ A(\mathrm{a}')}}{\vdots}\ \ A(\mathrm{b})\&\neg A(t),\Gamma,\pi\to\Delta,\Lambda,A(\mathrm{b}')\&\neg A(t)}{\Gamma,\ \pi\to\Delta,\ \Lambda,\ A(t)\&\neg A(t)}$$

3.16.2.2 The case OA is divided into three cases.

3.16.2.2.1 Only the left sub-formula of R is the chief-formula of 2.4. Let b be a variable not contained in Γ, Δ, $A(\mathrm{a})$, B.

Reduction is as follows ;

$$\dfrac{\dfrac{\overline{\Gamma\to\Delta,\ A(\mathrm{o})} \qquad \overline{\Gamma\to\Delta,\ \neg B}}{\vdots}\ \ \Gamma\to\Delta,\ \neg(\neg A(\mathrm{o})\vee B) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(\mathrm{a}),\ \Gamma\to\Delta,\ A(\mathrm{a}')}{\vdots}}{A(\mathrm{b}),\ \Gamma\to\Delta,\ A(\mathrm{b}')} \qquad \dfrac{\dfrac{B,\ \Gamma\to\Delta}{\Gamma\to\Delta,\ \neg B}}{A(\mathrm{b}),\ \Gamma\to\Delta,\ \neg B}}{A(\mathrm{b}),\ \Gamma\to\Delta,\ A(\mathrm{b}')\&\neg B}}{A(\mathrm{b})\&\neg B,\ \Gamma\to\Delta,\ A(\mathrm{b}')\&\neg B}}{\vdots}\ \ \neg(\neg A(\mathrm{b})\vee B),\ \Gamma\to\Delta.\ \neg(\neg A(\mathrm{b}')\Lambda B)}{\dfrac{\Gamma\to\Delta,\ \neg(\neg A(t)\vee B)}{\neg\neg(\neg A(t)\vee B),\ \Gamma\to\Delta}}$$

In this case a new NA yields again, and hence it needs considerations of more steps of the proof-figures.

3.16.2.2.2 Only the right sub-formula of R is the chief-formula of 2.4. Reduction is obtained similarly to 3.16.2.2.1.

3.16.2.2.3 Two sub-formulas of R are the chief-formulas of 2.4, and reduction is as follows ;

$$\dfrac{\dfrac{\dfrac{\Gamma\to\triangle,\ A(o)\qquad A(a),\ \Gamma\to\triangle,\ A(a')}{\Gamma\to\triangle,\ A(t)}}{\neg A(t),\ \Gamma\to\triangle}\qquad\dfrac{\dfrac{\Gamma\to\triangle, B(o)\qquad B(b),\Gamma\to\triangle, B(b')}{\Gamma\to\triangle,\ B(s)}}{\neg B(s),\ \Gamma\to\triangle}}{\neg A(t)\vee\neg B(s),\ \Gamma\to\triangle}$$

Since the reduction is simple, when t and s are generated by the same variable, and when at least one of them is a constant, we shall consider the case, where t and s are generated from different variables. Let c be a variable not contained A(a), B(b), Γ, $\triangle$, and d be a variable not contained in A(a), B(b), Γ, $\triangle$. Rediction is obtained as follows ;

$$\dfrac{\dfrac{\Gamma\to\triangle,\ A(o)\qquad \Gamma\to\triangle,\ B(o)}{\Gamma\to\triangle,\ \neg(\neg A(o)\vee\neg B(o))}\qquad \dfrac{\dfrac{\dfrac{A(a),\ \Gamma\to\triangle,\ A(a')\qquad \dfrac{\Gamma\to\triangle,\ B(o)}{B(o),\ \Gamma\to\triangle,\ B(o)}}{\downarrow}}{A(a)\&B(o),\ \Gamma\to\triangle,\ A(a')\&B(o)}}{\neg(\neg A(a)\vee\neg B(o)),\Gamma\to\triangle,\neg(\neg A(a')\wedge\neg B(o))}}{\Gamma\to\triangle,\ \neg(\neg A(t)\vee\neg B(o))}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma\to\triangle,\ A(o)\qquad B(b),\ \Gamma\to\triangle,\ B(b')}{B(d),\ \Gamma\to\triangle,\ A(o)\&B(d')}}{A(t)\&B(d),\ \Gamma\to\triangle,\ A(o)\&B(d')}}{\neg(\neg A)\,t)\vee\neg B(d)),\ \Gamma\to\triangle,\ \neg(\neg A(o)\vee\neg B(d))}}{\Gamma\to\triangle,\ \neg(\neg A(t)\vee B(o))}\qquad \dfrac{\dfrac{\dfrac{\dfrac{A(a),\ \Gamma\to\triangle,\ A(a')}{\vdots}\ \dfrac{B(b'),\ \Gamma\to\triangle,\ B(b')}{\vdots}}{A(c),\ \Gamma\to\triangle,\ A(c')\qquad B(d'),\ \Gamma\to\triangle,\ B(d')}}{\neg(\neg A(c)\vee\neg B(d'))\ \Gamma\to\triangle,\ \neg(\neg A(c')\vee\neg B(d')}}{\dfrac{\neg(\neg A(c)\vee\neg B(d')),\ \neg(\neg A(t)\vee\neg B(d)),\ \Gamma\to\triangle,\ \neg(\neg A(c')\vee\neg B(d'))}{\neg(\neg A(t)\vee\neg B(d)),\ \Gamma\to\triangle,\ \neg(\neg A(t)\vee B(d'))}}}{\dfrac{\Gamma\to\triangle,\ \neg(\neg A(t)\vee\neg B(s))}{\neg\neg(\neg A(t)\vee\neg B(s)),\ \Gamma\to\triangle}}$$

Here again NA yields, hence we must consider of more steps of the proof-figures.

3.16.2.3 The case UA is as follows ;

$$\dfrac{\dfrac{\dfrac{\Gamma\to\triangle,\ A(o)\qquad A(a),\ \Gamma\to\triangle,\ A(a')}{\Gamma\to\triangle,\ A(t)}}{\neg A(t),\ \Gamma\to\triangle}}{\neg A(t)\&B,\ \Gamma\to\triangle}$$

Let b, be variables not contained in $A(t)$, B, Γ, $\triangle$.
Reduction is given as follows ;

$$
\dfrac{\dfrac{\Gamma\to\triangle,\ A(o)}{\dfrac{\vdots}{\Gamma\to\triangle,\ \neg(\neg A(o)\,\&\,B)}} \qquad \dfrac{\dfrac{\dfrac{\neg B\to\neg B \qquad \dfrac{A(a),\ \Gamma\to\triangle,\ A(a')}{\dfrac{\vdots}{A(b),\ \Gamma\to\triangle,\ A(b')}}}{\cdots}}{A(b)\vee\neg B,\ \Gamma\to\triangle,\ A(b')\vee\neg B}}{\dfrac{\vdots}{\neg(\neg A(b)\,\&\,B),\ \Gamma\to\triangle,\ \neg(\neg A(b')\vee\neg B)}}}{\dfrac{\Gamma\to\triangle,\ \neg(\neg A(t)\,\&\,B)}{\neg\neg(\neg A(t)\,\&\,B),\ \Gamma\to\triangle}}
$$

The situation is the same as in 3.16.2.2.3.

3.16.2.4 the case EA, and 3.16.2.5 the case AA require no more investigation than in 3.11.

3.16.2.6 The case NS. Reduction is trivial.

3.16.2.7. The case FS.

Reduction is as follows ; Let b be a variables not contained in $A(a)$, B, Γ, $\triangle$.

$$
\dfrac{\dfrac{\Gamma\to\triangle,\ B,\ A(o)}{\dfrac{\vdots}{\dfrac{\Gamma\to\triangle,\ A(o)\vee B}{\dfrac{\downarrow}{\Gamma\to\triangle,\ \neg A(c)\supset B}}}} \qquad \dfrac{\dfrac{\dfrac{B\to B \qquad \dfrac{A(a),\ \Gamma\to\triangle,\ B,\ A(a')}{\dfrac{\vdots}{A(b),\ \Gamma\to\triangle,\ B,\ A(b')}}}{\cdots}}{A(b)\vee B,\ \Gamma\to\triangle,\ B,\ A(b')\vee B}}{\dfrac{\downarrow}{\dfrac{A(b)\vee B,\ \Gamma\to\triangle,\ A(b')\vee B}{\dfrac{\downarrow}{\neg A(b)\supset B,\ \Gamma\to\triangle,\ \neg A(b')\supset B}}}}}{\Gamma\to\triangle,\ \neg A(t)\supset B}
$$

3.16.2.8 The FA, as follows ;

$$
\dfrac{\Gamma\to\triangle,\ A \qquad \dfrac{\dfrac{\pi\to\Lambda,\ B(o) \qquad B(a),\ \pi\to\Lambda,\ B(a')}{\pi\to\Lambda,\ B(t)}}{\neg B(t),\ \pi\to\Lambda}}{A\supset\neg B(t),\ \Gamma,\ \pi\to\triangle,\ \Lambda}
$$

Let b be a variable not contained in A, $B(a)$, Γ, $\triangle$, π, Λ.

Reduction is obtained as follows ;

$$\dfrac{\dfrac{\Gamma\to\Delta,\ A}{\vdots}}{A,\ \Gamma,\ \pi\to\Delta,\ \Lambda,\ A}\qquad \dfrac{\dfrac{B(\mathrm{a}),\ \pi\to\Lambda,\ B(\mathrm{a}')}{\vdots}}{B(\mathrm{b}),\ \Gamma,\ \pi\to\Delta,\ \Lambda,\ B(\mathrm{b}')}$$

$$\dfrac{\dfrac{\Gamma\to\Delta,\ A \qquad \pi\to\Lambda,\ B(\mathrm{o})}{\Gamma,\pi\to\Delta,\Lambda,\neg(A\supset\neg B(\mathrm{o}))}\qquad \dfrac{\dfrac{A\&B(\mathrm{b}),\ \Gamma,\ \pi\to\Delta,\ \Lambda,\ A\&B(\mathrm{b}')}{\vdots}}{\neg(A\supset\neg B(\mathrm{b})),\Gamma,\pi\to\Delta,\Lambda,\neg(A\supset\neg B(\mathrm{b}'))}}{\dfrac{\Gamma,\ \pi\to\Delta,\ \Lambda,\ \neg(A\supset\neg B(t))}{\neg\neg(A\supset\neg B(t)),\ \Gamma,\ \pi\to\Delta,\ \Lambda}}\ \text{by 1.4}$$

By the above considerations, it is seen that NA is transferred to other inferences, or to a new inference NA. But in our case the end-sequent is such as, and hence NA must leave to other inferences at last.

3.17 The case FS, as follows ;

$$\dfrac{\dfrac{A,\ \Gamma\to\Delta,\ B(\mathrm{o}) \qquad B(\mathrm{a}),\ A,\ \Gamma\to\Delta,\ B(\mathrm{a}')}{A,\ \Gamma\to\Delta,\ B(t)}}{\Gamma\to\Delta,\ A\supset B(t)}$$

Reduction is obtained by the method in 3.16.2.7.

3.18 The case FA, as follows ;

$$\dfrac{\dfrac{\Gamma\to\Delta,\ A(\mathrm{o}) \qquad A(\mathrm{a}),\ \Gamma\to\Delta,\ A(\mathrm{a}')}{\Gamma\to\Delta,\ A(t)} \qquad B,\ \pi\to\Lambda}{A(t)\supset B,\ \Gamma,\ \pi\to\Delta,\ \Lambda}$$

Let c be a variable not contained in $A(\mathrm{a})$, B, Γ, Δ, π, Λ. Reduction is transferred to 2.16 as follows.

$$\dfrac{\dfrac{\dfrac{A(\mathrm{a}),\ \Gamma\to\Delta,\ A(\mathrm{a}')}{\vdots}}{\text{variable}}}{A(\mathrm{b}),\ \Gamma\to\Delta,\ A(\mathrm{b}')}\qquad \dfrac{B,\ \pi\to\Lambda}{\pi\to\Lambda,\ \neg B}$$

$$\dfrac{\dfrac{\Gamma\to\Delta,\ A(\mathrm{o}) \qquad \dfrac{B,\ \pi\to\Lambda}{\pi\to\Lambda,\ \neg B}}{\Gamma,\ \pi\to\Delta,\ \Lambda,\ \neg(A(\mathrm{o})\supset B)}\qquad \dfrac{\dfrac{A(\mathrm{b})\&\neg B,\ \Gamma,\ \pi\to\Delta,\ \Lambda,\ A(\mathrm{c}')\&\neg B}{\vdots}}{\neg(A(\mathrm{b})\supset B),\ \Gamma,\ \pi\to\Delta,\ \Lambda\ \neg(A(\mathrm{c}')\supset B)}}{\dfrac{\Gamma,\ \pi\to\Delta,\ \Lambda,\ \neg(A(t)\supset B)}{\neg\neg(A(t)\supset B),\ \Gamma,\ \pi\to\Delta,\ \Lambda}}\ \text{by 1.4}$$

Thus the inferences of induction can be put back by one step, and then we obtain our theorem.

4 Proof of independence of the formula (A) and its result.

4.1 In § 3 we obtain the way to reduce a proof-figure of an arbitrary normal formula of the form $(x_1)\,(x_n)\,A\,(x_1\cdots,x_n)$, and therefore, if a proof of a normal formula F of that form contains no cut, then in its normal proof all inductions are concerning only the variables of F. For example,

$$\downarrow\ \underline{F(u,b,c'),F(u,v,c)\to F(o,b',c')\ F(a'b',c')\ F(u,b,c'),\ F(u,v,c)\to F(a',b',c')}$$

$$\downarrow\ \underline{F(u,\ v,\ c)\to F(u,\ o,\ c)\ \ F(u,\ b,\ c')\to F(u,\ v,\ c)\to F(u,\ b',\ c')}$$

$$\underline{\to F(u,\ v,\ o)\ \ F(u,\ v,\ c)\to F(u,\ v,\ c')}$$

$$\to F(u,\ v,\ w)$$

where $F\ (a,\ b',\ c')$, $F\ (u,\ b,\ c')$, $F\,(u,\ v,\ c)\to F\ (a',\ b',\ c')$ are provable, without any inference of 2.4.

4.2 But it is not the case when cuts are contained in the proof-figure ; for, there may exist inferences of 2.4 with respect to the formulas which are eliminated by the cut, For instance, in 4.1,

$$F(a,\ b',\ c'),\ F(u,\ b,\ c'),\ F(u,\ v,\ c)\to F\ (a',\ b',\ c')$$

may be provable only by inferences of 2.4.

Now, if in a normal proof there is a inference of 2.4 with respect to the variable of the formula in the end-sequent above a inference of 2.4 concerning a formula eliminated by cut, reduction is as follows ;

$$\underline{B(b),\Gamma_1\to\triangle_1,B(b),\ A(c)\qquad A(a),B(b),\Gamma_1\to\triangle_1,B(b'),A(a')}$$

$$\underline{\Gamma\to\triangle,\ B(b),\ A(t)}$$

$$\downarrow\qquad\qquad \underline{\vdots}$$

$$\underline{\Gamma_1\to\triangle_1,\ (A\,t),\ B(c)\qquad B(b),\ \Gamma_1\to\triangle_1,\ A(t),\ B'(b')}$$

$$\underline{\Gamma_1\to\triangle_1,\ A(t),\ B(s)\qquad B(s),\ \pi\to\Lambda}$$

$$\Gamma_1,\ \pi\to\triangle_1,\ A(t),\ \Lambda$$

The proof is reducible to the following proof-figure ;

$$\downarrow$$

$$\underline{A(a),\ B(b),\ \Gamma_1\to\triangle_1,\ B(b'),\ A(a')}$$

$$\underline{\qquad\downarrow\qquad}$$

$$\underline{\Gamma_1\to\triangle_1,\ A(t),\ B(o)}\qquad \underline{A(c),\ B(b),\ \Gamma\to\triangle,\ B(b'),\ A(c')}$$

$$\downarrow\qquad\qquad\underline{\qquad\downarrow\qquad}$$

$$\underline{A(c),\Gamma_1\to\triangle_1,A(c'),A(t),\ B(o),B(d),A(c),\Gamma_1\to\triangle_1,A(c'),A(t),B(b')}$$

$$\underline{A(c),\ \Gamma_1\to\triangle_1,\ A(c'),\ A(t),\ B(s)\qquad B(s),\ \pi\to\Lambda}$$

$$A(c),\ \Gamma_1,\ \pi\to\triangle_1,\ ,\ \Lambda\,A(t),\ A(c')$$

$$\dfrac{\dfrac{\Gamma_1 \to \Delta_1,\ A(t),\ B(o)}{\Gamma_1 \to \Delta_1,\ A(t)A(o),\ B(o)} \qquad \dfrac{B(b),\ \Gamma_1 \to \Delta_1,\ B(b'),\ A(o)}{B(b),\ \Gamma_1 \to \Delta_1,\ A(t),\ A(o),\ B(b')}}{\dfrac{\Gamma_1 \to \Delta_1,\ A(t),\ A(o),\ B(s) \qquad B(s),\ \pi \to V}{\Gamma_1\ ;\ \pi \to \Delta_1,\ A(t),\ A(o),\ \Lambda}}$$

$$\dfrac{\Gamma_1\ ;\ \pi \to \Delta_1,\ \Lambda,\ A(t),\ A(o) \qquad A(c),\ \Gamma_1,\ \pi \to \Delta_1\ \Lambda,\ A(t),\ A(c')}{\Gamma_1,\ \pi \to \Delta_1,\ \Lambda,\ A(t),\ A(t)}$$

Where c is a variable not contained in $A(a)$, $A(t)$, $B(d)$, $B(s)$, Γ, Δ, π, Λ. By this reduction we can sharpen Theorem 1, that is, the following theorem is obtained.

Theorem. 2 For any normal formula of the form $(x_1)(x_2)\cdots(x_n)F(x_1, \cdots x_n)$ which is provable in LK_1, there is a normal proof, where all inferences of 2.4 with respect to the formula eliminated by cuts are above other inference of 2.4.

4.3 Before our independence-proof we shall prove several properties of the formula (A). 4.3.11.

Let $F(a, b)$ denote the formula $a < b \supset a' = b \vee a' < b$.

4.3.1 If $F(a, b) \to F(a', b)$ is provable in LK_1, then $\to F(a', b)$ is provable in LK_1.

Proof. By [2], it is enough to prove that if $F(a, b) \supset F(a, b)$ is provable in B_1, $F(a', b)$ is provable in B_1. Suppose that

4.3.1.1 $(a < b \supset a' = b \vee a' < b) \supset (a' < b \supset a'' = b \vee a'' < b)$

4.3.1.2 $\neg a < b \vee a' = b \vee a' < b \supset \neg a' < b \vee a'' \equiv b \vee a'' < d$.

4.3.1.3 $a' < b \supset a < b$

4.3.1.4 $\neg a < b \supset \neg a' < b$ 4.3.1.3

4.3.1.5 $\neg a < b \supset \neg a' < b \vee a'' = b \vee a''\ b$ 4.3.1.4

4.3.1.6 $a' = b \supset \neg a'\ b$ ( )

4.3.1.7 $a' = b \supset \neg a' < b \vee a'' = b \vee a'' \vee b$ 4.3.1.6

4.3.1.8 $a' < b \supset \neg a' < b \vee a'' = b \wedge a'' < b$ 4.3.1.2, 4.3.1.5, 4.3.1.7

4.3.1.9 $a' < b \supset a'' = b \vee a'' < b$ 4.3.1.8

4.3.2 If $F(a', b) \to F(a, b)$ is provable in LK_1, then $\to F(a, b)$ is provable in LK_1.

proof. Suppose that

4.3.2.1 $(a' < b \supset a'' = b \vee a'' < b) \supset (a < b \supset a' = b \vee a' < b)$

4.3.2.2 $\neg a' < b \vee a'' = b \vee a'' < b \supset \neg a < b \vee a' = b \vee a' < b$

4.3.2.3 $a'' = b \supset a' < b$

4.3.2.4 $a'' = b \supset \neg a < b \vee a' = b \vee a' < b$

4.3.2.5 $a'' < b \supset a' < b$

4.3.2.6 $a'' < b \supset \neg a < b \vee a' = b \vee a' < b$

4.3.2.7 $\neg a' < b \supset \neg a < b \vee a' = b \vee a' < b$

4.3.2.8 $\neg a<b \vee a'=b \vee a'<b$

Let LK_2 denote a system $LK_1+(<_4)$; where $(<_4)$ is the formula $a<b \supset a'<b'$.

4.33 If $F(a, b) \rightarrow F(a, b')$ is provable in LK_2, then $\rightarrow F(a, b')$ is provable in LK_2.

Proof.

Suppose that

4.3.3.1 $(a<b \supset a'=b \vee a'<b) \supset (a<b' \supset a'=b' \vee a'<b')$

4.3.3.2 $\neg a<b \vee a'=b \vee a'<b \supset \neg a<b' \vee a'=b' \vee a'<b'$

4.3.3.3 $a'=b \supset a'<b'$

4.3.3.4 $a'=b \supset \neg a<b' \vee a'=b \vee a'<b'$

4.3.3.5 $a'<b \supset a'<b'$

4.3.3.6 $a'<b \supset \neg a<b' \vee a'=b' \vee a'<b'$

4.3.3.7 $\neg a<b \supset \neg a<b' \vee a'=b' \vee a'<b'$

4.3.3.8 $\neg a<b' \vee a'=b' \vee a'<b'$

4.3.4 If $F(a, b) \rightarrow F(a, b)$ is provable in LK_2, then $\rightarrow F(a, b)$ is provable in LK_2.

Proof. Suppose

4.3.4.1 $(a<b' \supset a'=b' \vee a'<b') \supset (a<b \supset a'=b \vee a'<b)$

4.3.4.2 $\neg a<b' \vee a'=b' \vee a'<b' \supset \neg a<b \wedge a'=b \vee a'<b$

4.3.4.3 $a<b \supset a<b'$

4.3.4.4 $\neg a<b' \supset \neg a<b \vee a'=b \vee a'<b$

4.3.4.5 $a'=b' \supset \neg a'<b'$

4.3.4.6 $a'=b' \supset \neg a<b \vee a'=b \vee a'<b$

4.3.4.7 $a'<b' \supset \neg a<b \vee a'=b \vee a'<b$

4.3.4.8 $\neg a<b \vee a'=b \vee a'<b$

4.3.5 If $F(a, b) \rightarrow F(a_{(n)}, b)$ is provable in LK_1, then $F(a_{(n)}, b)$ is provable in LK_1.

Proof. If $n=0$, proof is evident.

Suppose

4.3.5.1 $(a<b \supset a'=B \vee a'<b) \supset (a^{(n)}<b \supset a^{(n+1)}=b \vee a^{(n+1)}<b)$

4.3.5.2 $a^{(n)}<b \supset a<b$

5.3.5.3 $\neg a<b \supset \neg a^{(n)}<b \vee a^{(n+1)}=b \vee a^{(n+1)}<b$

4.3.5.4 $a'=b \supset \neg a^{(n)}<b$

4.3.5.5 $a'=b \supset \neg a^{(n)}<b \vee a^{(n+1)}=b \vee a^{(n+1)}<b$

4.3.5.6 $a'<b \supset \neg a^{(n)}<b \vee a^{(n+1)}=b \vee a^{(n+1)}<b$

4.3.5.7 $a^{(n)}<b \supset a'<b$

4.3.5.8 $a^{(n)}<b \supset a^{(n+1)}=b \vee a^{(n+1)}<b$

4.3.6 If $F(a^{(n)}, b) \rightarrow F(a, b)$ is provable in LK_1, then $\rightarrow F(a, b)$ is provable in LK_1.

Proof. If $n=0$, proof is evident.

Suppose

4.3.6.1 $\neg a^{(n)} < b \vee a^{(n+1)} = b \vee a^{(n+1)} < b \supset \neg a < b \vee a' = b \vee a' < b$

4.3.6.2 $a^{(n+1)} = b \supset a' < b$

4.3.6.3 $a^{(n+1)} = b \supset \neg a < b \vee a' = b \vee a' < b$

5.3.6.4 $a^{(n+1)} < b \supset \neg a < b \vee a' = b \vee a' < b$

4.3.6.5 $\neg a^{(n)} < b \supset \neg a < b \vee a' = b \vee a' < b$

4.3.6.6 $\neg a < b \vee a' = b \vee a' < b$

4.3.7 If $F(a, b) \to F(a, b^{(n)})$ is provable in LK_2, then $\to F(a, b^{(n)})$ is prvoable in LK_2.

Proof. The case $n = 0$ is trivial.

4.3.7.1 $\neg a < b \vee a' = b \vee a' < b \supset \neg a < b^{(n)} < a' = b^{(n)} \vee a' < b^{(n)}$

4.3.7.2 $a' = b \supset a' < b^{(n)}$

4.3.7.3 $a' = b \supset \neg a < b^{(n)} \vee a' = b^{(n)} \vee a' < b^{(n)}$

4.3.7.4 $a' < b \supset \neg a < b^{(n)} \vee a' = b^{(n)} \vee a' < b^{(n)}$

4.3.7.5 $\neg a < b \supset \neg a < b^{(n)} \vee a' = b^{(n)} \vee a' < b^{(n)}$

4.3.7.6 $a < b \supset a' < b^{(n)}$

4.3.7.7 $\neg a < b^{(n)} \vee a' = b^{(n)} \vee a' < b^{(n)}$

4.3.8 If $F(a, b^{(n)}) \to F(a, b)$ is provable in LK_2, then $\to F(a, b)$ is provable in LK_2.

Proof ; $n = 0$, proof is trivial.

Suppose $\neg a < b^{(n)} \vee a' = b^{(n)} \vee a' < b^{(n)} \supset \neg a < b \vee a' = b < a' < b$

4.3.8.1 $a < b \supset a < b^{(n)}$

4.3.8.2 $\neg a < b^{(n)} \supset \neg a < b \vee a' = b \vee a' < b$

4.3.8.3 $a' = b^{(n)} \supset \neg a < b \vee a' = b \vee a' < b$

4.3.8.4 $a' < b^{(n)} \supset \neg a < b \vee a' = b \vee a' < b$

4.3.8.5 $a' < b^{(n)} \supset \neg a < b \vee a' = b \vee a' < b$

4.3.8.6 $\neg a < b \vee a' = b \vee a' < b$

4.3.9 If $F(a^{(m)}, b) \to F(a^{(m)}, b)$ is provable in LK_1, then $\to F(a^{(m)}, b)$ is provable in LK_1.

Proof. 4.3.5—6.

4.3.10. If $F(a, b^{(m)}) \to F(a, b^{(m)})$ is provable in LK_2, then $\to F(a, b^{(m)})$ is provable in LK_2.

4.3.11 $F(a, b) \to F(a^{(n)}, b^{(n)})$ is provable in LK_2.

4.4 If $F(a, b)a$, $a < b \supset a' = b \vee a' < b$, is provable, then there is the following normal proof.

$$\dfrac{\dfrac{A_2(a_2)\cdots\rightarrow\cdots A_2(a'_2),\ A_1(o) \qquad A_1(a_1),\ \cdots,\ A_n(a_n),\ F(a,\ b'),\ F(u,\ b)\rightarrow F(a,\ b'),\ A_n(a_n')\cdots A_1(a'_1)}{A_2(a_2)\ \cdots\ F(a,\ b'),\ F(u,\ b)\rightarrow F(a'_1\, b')\cdots A_2(a_2'),\ A_1(t_1)} \qquad A_1(t_1),\ \cdots\rightarrow\cdots}{A_2(a_2)\ \cdots F(u,\ b)\rightarrow F(a',\ b')\ \cdots\ A_2(a'_2)}$$

$$\dfrac{\dfrac{F(u,\ b)\rightarrow F(o,\ b')}{\rightarrow F(u,\ o)} \qquad \dfrac{F(a,\ b'),\ F(u,\ b)\rightarrow F(a',\ b')}{F(u,\ b)\rightarrow F(u,\ b')}}{\rightarrow F(u,\ v)}$$

or

$$\dfrac{\dfrac{F(a,\ v)\rightarrow F(a',\ o)}{\rightarrow F(o,\ v)} \qquad \dfrac{F(a',\ b),\ F(a,\ v)\rightarrow F(a',\ b')}{F(a,v)\rightarrow F(a',\ v)}}{\rightarrow F(u,\ v)}$$

Proof.

The formula A derived by inferences of induction generates a formula B (eventually A itself), which is eliminated by cut or generate the end-formula. We shall consider only the case where B is eliminated.

As is seen in §3, we have to consider only the case, where chief-formulas of inferences of induction is eliminated as follows ;

$$\cfrac{\cfrac{\cfrac{\Gamma \to \triangle,\ A(o) \qquad A(a),\ \Gamma \to \triangle,\ A(a')}{\Gamma \to \triangle,\ A(t)} \qquad A(t),\ \pi \to \Lambda}{\Gamma,\ \pi \to \triangle,\ \Lambda}}{\phi \to \psi}$$

4.4.0.1. When Γ, π, $\triangle$, Λ don't contain a and variables included in $A(t)$, the reduction is as follows ;

$$\cfrac{\cfrac{\cfrac{\Gamma \to \triangle,\ A(o)}{\Gamma,\ \pi \to \triangle,\ V,\ A(o)} \downarrow \atop \phi \to \psi,\ A(o) \qquad \cfrac{A(a),\ \Gamma \to \triangle,\ A(a')}{A(a),\ \Gamma,\ \pi \to \triangle,\ \Lambda,\ A(a')} \downarrow \atop A(a),\ \phi \to \psi,\ A(a)}{\phi \to \psi,\ A(t)} \qquad \cfrac{A(t),\ \pi \to \Lambda}{A(t),\ \Gamma,\ \pi \to \triangle,\ \Lambda} \downarrow \atop A(t),\ \phi \to \psi}{\phi \to \psi}$$

4.4.0.2. Γ, $\triangle$ contain a variable included in $A(t)$, but π, Λ do not so.

$$\Gamma,\ \pi \to \triangle,\ \Lambda$$
$$\downarrow$$
$$\phi \to \psi$$

It is evident when the variable included in $A(t)$ is no eigenvariable of inference 2.4.

$$\cfrac{\cfrac{\cfrac{\Gamma \to \triangle,\ A(o)}{\vdots}}{\Gamma,\ \pi \to \triangle,\ \Lambda,\ A(o)} \downarrow \atop \phi \to \psi,\ A(o) \qquad \cfrac{\cfrac{A(a),\ \Gamma \to \triangle,\ A(a')}{\vdots}}{A(a),\ \Gamma,\ \pi \to \triangle,\ \Lambda,\ A(a')} \downarrow \atop A(a),\ \phi \to \psi,\ A(a')}{\phi \to \psi,\ A(t) \qquad\qquad \cfrac{\cfrac{A(t),\ \pi \to \Lambda}{\vdots}}{A(t),\ \Gamma,\ \pi \to \triangle,\ \Lambda} \downarrow \atop A(t),\ \phi \to \psi}$$
$$\phi \to \psi$$

We shall consider the case, where a variable included in $A(t)$ has to satisfy the variable-condition as the eigenvariable of the inference.

$$
\begin{array}{c}
\Gamma, \ \pi \rightarrow \triangle, \ A \\
\downarrow \\
\dfrac{\phi_{0} \rightarrow \psi_{0}, \ B(b)}{\phi_{0} \rightarrow \psi_{0}, \ (x)B(x)} \\
\downarrow \\
\phi \rightarrow \psi
\end{array}
\qquad\qquad
\begin{array}{c}
\Gamma, \ \pi \rightarrow \triangle, \ A \\
\downarrow \\
\dfrac{B(b), \ \phi \rightarrow \psi_{0}}{(\exists x)B(x), \ \phi_{0} \rightarrow \psi_{0}} \\
\downarrow \\
\phi \rightarrow \psi
\end{array}
$$

Every inference of induction can be put off one by one, andthen only eigenvariables of (x), $(\exists x)$ need to be considered. The end-formula is of the form $A(a, b, \cdots)$, and then $(x)B(x)$ must generate a formula D, which is eliminated by cut.

$$
\begin{array}{c}
\begin{array}{cc}
\begin{array}{c} \phi_{0} \rightarrow \psi_{0}, \ (x)B(x) \\ \downarrow \end{array} & \\
\end{array} \\
\dfrac{\phi_{1} \rightarrow \psi_{1}, \ D \qquad\qquad D_{1} \ \phi_{2} \rightarrow \psi_{2}}{\phi_{1}, \ \phi_{2} \rightarrow \psi_{1}, \ \psi_{2}} \\
\downarrow \\
\phi \rightarrow \psi
\end{array}
$$

or

$$
\begin{array}{c}
\begin{array}{cc}
 & \begin{array}{c} \phi_{0} \rightarrow \psi_{0}, \ (x)B(x) \\ \downarrow \end{array} \\
\end{array} \\
\dfrac{\phi_{2} \rightarrow \psi_{2}, \ D \qquad\qquad D, \ \phi_{1} \rightarrow \psi_{1}}{\phi_{2}, \ \phi_{1} \rightarrow \psi_{2}, \ \psi_{1}} \\
\downarrow \\
\phi \rightarrow \psi
\end{array}
$$

D contains $(x)B(x)$ as its part, then

$$
\begin{array}{c}
\downarrow \\
\underline{(x)B(x), \ \phi_{3} \rightarrow \psi_{3}} \\
\downarrow \\
D, \ \phi_{2} \rightarrow \psi_{2}
\end{array}
$$

is a proof of $D, \ \phi_{2} \rightarrow \psi_{2}$ Therefore

4.4.0.2.1. $(x)B(x)$ is derivable by the proof

$$\begin{array}{c} \downarrow \\ B(a),\ \phi_4 \to \psi_4 \\ \hline (x)B(x),\ \phi_4 \to \psi_4 \\ \downarrow \\ (x)B(x),\ \phi_3 \to \psi_3 \end{array}$$

or by

4.4.0.2.2.

$$\begin{array}{c} \phi_4 \to \psi_4 \\ \hline (x)B(x),\ \phi_4 \to \psi_4 \end{array}$$

or by

4.4.0.2.3.

$$\begin{array}{c} (x)B(x) \to (x)B(x) \\ \downarrow \\ \hline (x)B(x),\ \phi_4 \to \psi_4 \end{array}$$

Reduction in the cases 4.4.0.2.1—2. is as follows ;

$$\begin{array}{c} \Gamma \to \triangle,\ A(o) \\ \hline \vdots \\ \hline \Gamma, \pi \to \triangle, \Lambda, A(o) \end{array} \qquad \begin{array}{c} A(a), \Gamma \to \triangle,\ A(a') \\ \hline \vdots \\ \hline A(a), \Gamma, \pi \to \triangle, \Lambda, A(a') \end{array}$$

$$\begin{array}{c} \downarrow \qquad\qquad \dfrac{}{\phi_4 \to \psi_4} \qquad\qquad \downarrow \qquad\qquad \downarrow \\ \dfrac{\phi_o \to \psi_o . A(o), B(b) \quad B(b), \phi_4 \to \psi_4}{\phi_o,\ \phi_4 \to \psi_o, A(o), \psi_4} \qquad \dfrac{\dfrac{A(a). \phi_o \to \psi_o A,(a'), B(b) \quad \dfrac{}{B(b), \phi_4 \to \psi_4}}{A(a),\ \phi_o, \phi_4 \to \psi_o, A(a'), \psi_4}}{\vdots} \qquad \dfrac{\phi_4 \to \psi_4}{} \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \dfrac{A(t), \pi \to \Lambda}{\downarrow} \\ \hline \dfrac{\phi_o, \phi_4 \to \psi_o, \psi_4, A(o) \quad A(a), \phi_o \phi_4 \to \psi_o . \psi_4, A(a')}{\phi_o,\ \phi_4 \to \psi_o,\ \psi_4,\ A(t)} \qquad \dfrac{A(t) . \phi_o \to \psi_o, B \quad B, \phi_4 \to \psi_4}{A(t),\ \phi_o,\ \phi_4 \to \psi_o,\ \psi_4} \\ \hline \phi_o,\ \phi_4 \to \psi_o,\ \psi_4 \\ \downarrow \\ \phi \to \psi \end{array}$$

If eigenvariables in Γ, π, $\triangle$, Λ are exchanged by suitable variables, the above proof of $\phi \rightarrow \psi$ from ϕ_o, $\phi_4 \rightarrow \psi_o$, ψ_4 exactly exists.

In the case 4.4.0.2.3. $(x)B(x)$ must be eliminated, and there must be the following in the proof, and then consideration analogous to the above , concerning $\phi_o \rightarrow \psi_o$, $B(a)$ and E, $\phi_5 \rightarrow \psi_4$ is sufficient.

$$(x)B(x) \rightarrow (x)\ B(x)$$

$$\downarrow$$

$$\frac{(x)B(x),\ \phi_4 \rightarrow \psi_4\ E \qquad E,\ \phi_5 \rightarrow \psi_5}{(x)B(x),\ \phi_4,\ \phi_5 \rightarrow \psi_4,\ \psi_5}$$

Thus our theorem is proved.

Now, let $L\ K_3$ be a system consisting of (J_1), (J_2), $(<_1)$, $(<_2)$, $(<_3)$, $(<_4)$, $(<_5)$, $(<_6)$.

$(<_5)$ $\quad \neg 0 = a \supset 0 < a$

$(<_6)$ $\quad (0 =^{(n)} a \vee 0^{(n)} < a) \supset (\exists x)(x' = a)$ for $n \neq o$

$L\ K_3$ is equivalent to the system (A) in [I] p. 263, then the completeness theorem in [I] p. 264 is effected.

In the above proof

$$A_1(a_1) \cdots F(a', b'),\ F(u, b) \rightarrow F(a', b'),\ \cdots\cdots,\ A_1(a'_1)$$

is provable without inferences of 2.4. Therefore,

$$A_1(a_1) \cdots F(u, b) \rightarrow F(a', b') \cdots A_2(a'_2)$$

is provable in $L\ K_3$, or

$$A_1(a_1) \cdots \rightarrow A_1(a'_1)$$

is provable $L\ K_3$.

Now, suppose that

$$A_2(a_2) \cdots \rightarrow \cdots A_2(a'_2)$$

is not provable without the inference of 2.4 with respect to $A_1(a_1)$.

4.4.1.1 If

$$A_1(a_1) \cdots \rightarrow F(a', b') \cdots A_2(a'_2)$$

is provable in $L\ K_3$, then another proof of

$$A_2(a_2) \cdots \rightarrow A_2(a_2')$$

is obtained as follows ;

$$\frac{A_2(a_2) \cdots \rightarrow \cdots A_2(a'_2),\ A_1(o) \qquad \dfrac{A_1(a_1),\ A_2(a_2) \cdots \rightarrow \cdots A_2(a'_2)}{A_1(o),\ A_2(a_2) \cdots \rightarrow \cdots A_2(a'_2)}}{A_2(a_2) \cdots \rightarrow \cdots A_2(a'_2)}$$

This contradicts the hypothesis, and then

4.4.1.2 $A_1(a_1) \cdots \rightarrow A_1(a'_1)$

is provable in $L\ K_3$. Therefore,

$$A_1(a_1),\ F(a, b'),\ F(u, b) \rightarrow A_1(a'_1)$$

is provable without the above mentioned inference 2.4. But, evidently,

$F(a, b'), F(u, b) \to$

is not provable, then we have

$F(a, b'), F(u, b) \to A_1(a)$

in $L K_1$. a theorem in

Therefore, by [1], (p. 264),

$(x)(y)F(x, y) \to A_1(a)$

is provable in $L K_3$.

That is, there are variable terms $a'' \cdots, a_{1k1}, b'', \cdots, b_{1k1}$

such that $F(a'', b'') \cdots F(a_{1k1}, b_{1k1}) \to A_1(a)$

is provable in $L K_3$.

Therefore,

$A_2(t_2) \cdots A_n(t_n)\ F(a'', b'') \cdots F(a_{1k1}, b_{1k1}), F(a, b'), F(u, b) \to F(a', b')$

is provable without the inference of 2.4 with respect to $A_1(a)$.

Finally, at least one of two sequents

$A_2(a_2) \cdots \to A_2(a_2')$

or

$A_2(t_2) \to A_2(t_2')$

is provable without the infrence of 2.4 with respect to $A_1(a)$.

If we go on these considerations, we have the followineg proposition.

There are variable terms $a_1, \cdots, a_l, b_1, \cdots b_l$

such that

$F(a_1, a_1) \cdots F(a_l, b_l), F(a, b'), F(u, b) \to F(a', b')$

is provable in LK_0.

But, by [2], the sequent is provable without cut, because it is provable in $L K_3$.

4.4.2 We shall define "number" as follows ;

4.4.2.1.1 o, w are numbers.

4.4.2.1.2 If a is a number, then a′ is a number.

4.4.2.1.3 The only numbers are given by 4.4.2.1.1—2.

The order $<$ between numbers is defined a sfollows ;

4.4.2.2.1 $a < a'$

4.4.2.2.2 If $a < b$, $b < c$, then $a < c$

4.4.2.2.3 $a < a$ is false.

4.4.2.2.4 $o^{(n)} < w^{(n)}$

4.4.2.2.5 $o^{(l)} = w^{(n)}$ is false

4.4.2.2.6 $o^{(n)} < w^{(m)}$ $(n > m)$, $w^{(n)} < o^{(m)}$are false

Numbers, order defined here, underestanding logical symbols in its ordinary meaning costruct a model, which satisfies B'_1 and $(<_4)$ $(<_5)$, $(<_6)$. If a_i is generated from a, and b_i from b, and

$F(a_i, b_i), \Gamma \to F(a', b')$

is provable in $L K_3$, then

$\Gamma \to F(a^{(n)}, b^{(n)})$

is provable in $L\ K_3$. And then if (A) is provable in $L\ K_1$,

$$F(a_1,\ b_1),\ F(u,\ b) \rightarrow F(a^{(n)},\ b^{(n)}) \quad \text{for any} \leftarrow n$$

is prvable in $L\ K_3$. But

$$F(a_1,\ a_1) \cdots,\ F(u,\ b) \rightarrow F(a^{(n)},\ b^{(n)})$$

is not universally true in M,

For, let $a^{(n)}$ be $o^{(k)}$ and $b^{(k)}$ be $w^{(n)}$, where k is large enough to be such that all variable terms in the antecedent, which is genereated from a, are $a^{(l)}, l>m$, where other variables are $w^{(m_0+i)}$, $i=o,\ 1,\ 2,\ \cdots$

$F(o^{(l)},\ w^{(m)}$is true $F(w^{(m)},\ o^{(n)}$ is true and $F(w^{(m)},\ w^{(l)})$ is true. But

$F(o^{(k)},\ w^{(k)})$ is false, because $o^{(k)} < w^{(k)}$ but $o^{(k+1)} = w^{(k)}$, $o^{(k+1)} < w^{(k)}$ are false.

And then

$$F(a_1,\ b_1) \cdots \rightarrow F(a^{(n)},\ b^{(n)})$$

is not provable in $L\ K_3$. By this proof, (A) is independent in $L\ K_1$. Thus, the independence of (A) is proved.

4.5 Consequence of the independence of (A). We shall prove that (B) is independent in $L\ K_2$.

$$(B)\ A(b)\&(x)(A(x) \supset A(x')) \supset (a=b \vee b<a \supset A(a))$$

4.5.1 Definition $F(u) \equiv b'=u \vee b'<u \vee b'=u'$

4.5.2 $F(b)$

4.5.3 $b'=a \supset F(a')$

4.5.4 $b'<a \supset F(a')$

4.5.5 $b'=a' \supset F(a')$

4.5.6 $F(a) \supset F(a')$

4.5.7 $F(b)\&(x)(F(a) \supset F(a'))$

Suppose (B) is provable.

4.5.8 $a=b \vee b<a \supset b'=a \vee b'<a \vee b'=a'$

4.5.9 $a=b \supset F(a)$

4.5.10 $b<a \supset b'=a \vee b'<a \vee b'=a'$

4.5.11 $b<a \supset b'<a'$

4.5.12 $b<a \supset \neg b'=a'$

4.5.13 $b<a \supset b'=a \vee b'<a$

But 4.5.13 is not provable in $L\ K_2$, and then (B) is independent in $L\ K_2$.

References.

1) [1] D. Hilbert u. P. Bernays, Grundlagen der Mathematik, vol. **1** (1934)

2) K. Gödel, Über formal unentscheldbare Sätze der Principia Mathematica und verwandter Syseme 1, Monats. fur Math. u. Phy. vol. **38**.pp. 173—198

E. Mendelson, Some proof of independence in axomatic set theory, J. Symb. Log. vol. **21**(1956) pp. 291—303.

3) [2] G. Gentzen, Untersuchungen über das logische Schliessen, I, II, Math. Z. vol. **39** (1935) pp. 176—210, 405—431.